Optimization of Chromatic Optics in Circular Accelerators

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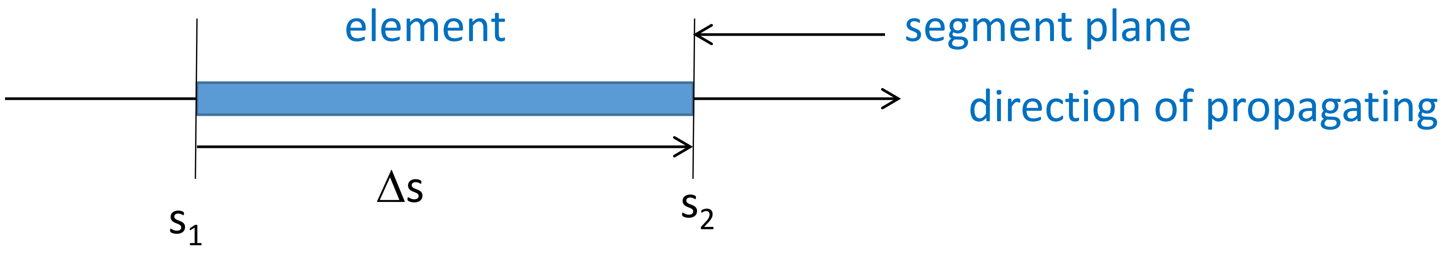
Concept of Transfer Map



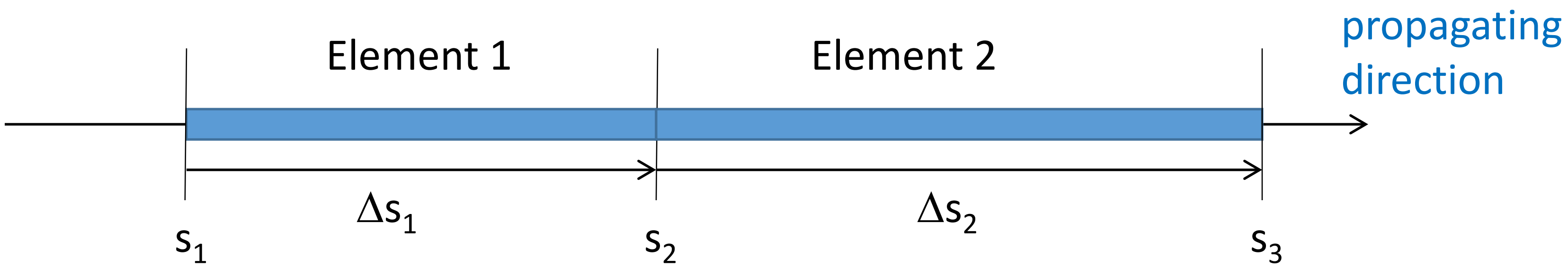
$$z(s_1) = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ l \end{pmatrix}\Bigg|_{s_1} \quad \mathcal{M}_{1\to 2} \quad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ l \end{pmatrix}\Bigg|_{s_2} = z(s_2)$$

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)).$$

abbreviated map notation

A set (six) of functions of canonical coordinates. It's called symplectic if its Jacob is symplectic.

Concatenation of Maps



If we have the transfer map for each individual elements:

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)),$$
$$z(s_3) = \mathcal{M}_{2\to 3}(z(s_2)).$$

Then the transfer map for the combined elements is given by

$$z(s_3) = \boxed{\mathcal{M}_{1\to 2} \circ \mathcal{M}_{2\to 3}}(z(s_1)) \equiv \mathcal{M}_{2\to 3}(\mathcal{M}_{1\to 2} z(s_1)),$$

$\mathcal{M}_{1\to 3}$

nested functions

Property of Symplectic Maps

Jacobian of a map:

$$J(\mathcal{M})=\begin{pmatrix}
\frac{\partial \mathcal{M}_1}{\partial x} & \frac{\partial \mathcal{M}_1}{\partial p_x} & \frac{\partial \mathcal{M}_1}{\partial y} & \frac{\partial \mathcal{M}_1}{\partial p_y} & \frac{\partial \mathcal{M}_1}{\partial \delta} & \frac{\partial \mathcal{M}_1}{\partial l} \\
\frac{\partial \mathcal{M}_2}{\partial x} & \frac{\partial \mathcal{M}_2}{\partial p_x} & \frac{\partial \mathcal{M}_2}{\partial y} & \frac{\partial \mathcal{M}_2}{\partial p_y} & \frac{\partial \mathcal{M}_2}{\partial \delta} & \frac{\partial \mathcal{M}_2}{\partial l} \\
\frac{\partial \mathcal{M}_3}{\partial x} & \frac{\partial \mathcal{M}_3}{\partial p_x} & \frac{\partial \mathcal{M}_3}{\partial y} & \frac{\partial \mathcal{M}_3}{\partial p_y} & \frac{\partial \mathcal{M}_3}{\partial \delta} & \frac{\partial \mathcal{M}_3}{\partial l} \\
\frac{\partial \mathcal{M}_4}{\partial x} & \frac{\partial \mathcal{M}_4}{\partial p_x} & \frac{\partial \mathcal{M}_4}{\partial y} & \frac{\partial \mathcal{M}_4}{\partial p_y} & \frac{\partial \mathcal{M}_4}{\partial \delta} & \frac{\partial \mathcal{M}_4}{\partial l} \\
\frac{\partial \mathcal{M}_5}{\partial x} & \frac{\partial \mathcal{M}_5}{\partial p_x} & \frac{\partial \mathcal{M}_5}{\partial y} & \frac{\partial \mathcal{M}_5}{\partial p_y} & \frac{\partial \mathcal{M}_5}{\partial \delta} & \frac{\partial \mathcal{M}_5}{\partial l} \\
\frac{\partial \mathcal{M}_6}{\partial x} & \frac{\partial \mathcal{M}_6}{\partial p_x} & \frac{\partial \mathcal{M}_6}{\partial y} & \frac{\partial \mathcal{M}_6}{\partial p_y} & \frac{\partial \mathcal{M}_6}{\partial \delta} & \frac{\partial \mathcal{M}_6}{\partial l}
\end{pmatrix}$$

constant J matrix:

$$J=\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}$$

Symplectic condition:

$$J(\mathcal{M})\bullet J\bullet J(\mathcal{M})^T=J$$

Specifically, R-matrix is given by $J(M)|_{x=px=y=py=\delta=l=0.}$ So it is sympletic as well.

Hamiltonian and Transfer Map for a Sector Bend Magnet

Use s as the independent variable, Hamiltonian in the paraxial approximation is given by,

$$H_{Sbend} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x}{\rho}\delta.$$

Solving the Hamiltonian equations, we obtain the transfer map of a sector bend:

$$\mathcal{M}_1 = x + \frac{L}{1+\delta}(p_x + \frac{\theta\delta}{2}),$$

$$\mathcal{M}_2 = p_x + \theta\delta,$$

$$\mathcal{M}_3 = y + \frac{Lp_y}{1+\delta},$$

$$\mathcal{M}_4 = p_y,$$

$$\mathcal{M}_5 = \delta,$$

$$\mathcal{M}_6 = l + \theta x + \frac{L}{2(1+\delta)^2}[p_x^2 + p_y^2 + \theta(1+2\delta)(p_x + \frac{\theta\delta}{3})],$$

where L is the length and $\theta = L/\rho$ the bending angle of the dipole.

Transfer Map for Thin Quadrupole and Sextupole

Transfer map is given by a kick:

$$\mathcal{M}_1 = x,$$

$$\mathcal{M}_2 = p_x - \frac{x}{f} - \frac{\kappa}{2}(x^2 - y^2),$$

$$\mathcal{M}_3 = y,$$

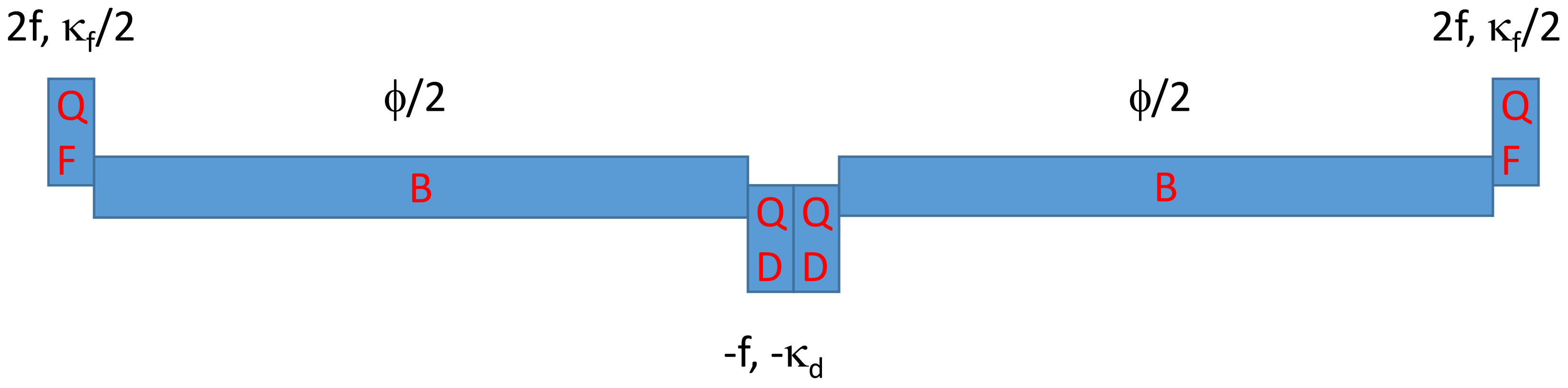
$$\mathcal{M}_4 = p_y + \frac{y}{f} + \kappa xy,$$

$$\mathcal{M}_5 = \delta,$$

$$\mathcal{M}_6 = l,$$

where f is the focusing (in horizontal plane) length of quadrupole and κ is the integrated strength of sextupole.

Periodic Cell: FODO



How to compute the Courant-Synder parameters and dispersions? For simplicity, we can use thin-lens approximation for quadrupoles, and small angle approximation for dipoles, and no gaps between any magnets.

What's the problem if we use these FODO cells to build entire ring? Why do we need to introduce sextupole magnets? How they work? Can we do better?

Courant-Snyder Parameters

Matrix of periodic system:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

$$M = ARA^{-1}$$

where A^{-1} is a transformations from physical to normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

A is an "ascript" and is not unique. Since two-dimensional rotational group is commutative AR(q) is also an ascript. Courant and Synder choose to have $A_{12}=0$.

Linear Optics

Using the transfer map of the cell and the R-matrix, we find that the betatron phase advances in both planes are the same $\mu_x=\mu_y=\mu$ and given by,

$$\sin\frac{\mu}{2} = \frac{L}{4f},$$

where L is the cell length. The beta functions at the beginning:

$$\beta_x = \frac{L(1+\sin\frac{\mu}{2})}{\sin\mu}, \beta_y = \frac{L(1-\sin\frac{\mu}{2})}{\sin\mu},$$

and the periodical dispersion:

$$\eta_0 = \frac{L\phi(1+\frac{1}{2}\sin\frac{\mu}{2})}{4\sin^2\frac{\mu}{2}}.$$

No surprises. They agree with the well-known results.

To the first-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$\mathcal{M}_{\eta 0} = \mathcal{A}_{\eta 0} \circ \mathcal{M}_{cell} \circ \mathcal{A}^{-1}_{\eta 0},$$

where the dispersive map is given by,

$$\begin{aligned}
\mathcal{A}_1 &= x + \eta_0 \delta, \\
\mathcal{A}_2 &= p_x, \\
\mathcal{A}_3 &= y, \\
\mathcal{A}_4 &= p_y, \\
\mathcal{A}_5 &= \delta, \\
\mathcal{A}_6 &= l - \eta_0 p_x,
\end{aligned}$$

Introducing a Jacobian operator, we find the matrix with dependence of δ:

$$R_{\eta 0}(\delta) = \mathcal{J}[\mathcal{M}_{\eta 0}] \equiv J(\mathcal{M}_{\eta 0})\,|_{x=px=y=py=l=0}$$

Like the R-matrix, it is symplectic.

Linear Chromaticity

Betatron phase advances up to the first-order of δ:

$$\mu_x(\delta) = \mu - \tan\frac{\mu}{2}[2 - \frac{1}{4\sin\frac{\mu}{2}}(\frac{1}{2} + \frac{1}{\sin^2\frac{\mu}{2}})(\kappa_f - \kappa_d) fL\phi - \frac{3}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d) fL\phi]\delta,$$

$$\mu_y(\delta) = \mu - \tan\frac{\mu}{2}[2 - \frac{1}{4\sin\frac{\mu}{2}}(\frac{1}{2} - \frac{1}{\sin^2\frac{\mu}{2}})(\kappa_f - \kappa_d) fL\phi - \frac{1}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d) fL\phi]\delta,$$

where κ_f,κ_d are the integrated strengths of the sextupoles. We can set their values:

$$\kappa_f = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1 + \frac{1}{2}\sin\frac{\mu}{2})}, \kappa_d = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1 - \frac{1}{2}\sin\frac{\mu}{2})},$$

to cancel the linear chromaticities in both planes. The settings are expected for the local compensation to the chromatic errors by quadrupoles.

To the second-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$\mathcal{M}_{\eta 1} = \mathcal{A}_{\eta 1} \circ \mathcal{A}_{\eta 0} \circ \mathcal{M}_{cell} \circ \mathcal{A}^{-1}_{\eta 0} \circ \mathcal{A}^{-1}_{\eta 1},$$

where the new dispersive map is given by,

$$\mathcal{A}_1 = x + \eta_1 \frac{\delta^2}{2},$$

$$\mathcal{A}_2 = p_x,$$

$$\mathcal{A}_3 = y,$$

$$\mathcal{A}_4 = p_y,$$

$$\mathcal{A}_5 = \delta,$$

$$\mathcal{A}_6 = l - \eta_1 p_x \delta,$$

where the first-order dispersion is found in the same way as the zeroth-order one, we have

$$\eta_1 = -\frac{f\phi}{2}.$$

Using the Jacobian operator, we find the matrix with dependence of δ:

$$R_{\eta 1}(\delta) = \mathcal{J}[\mathcal{M}_{\eta 1}]$$

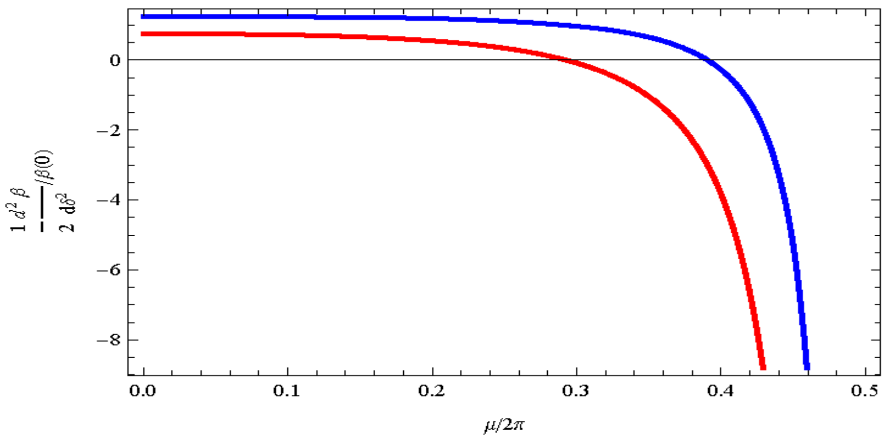
Like the R-matrix, it is symplectic.

Second-Order Beta Beating

The beta functions at the beginning of the cell, up to the second-order of δ:

$$\beta_x(\delta) = \beta_x[1 - \delta + \frac{10 - 13\sin^2\frac{\mu}{2} - \sin^3\frac{\mu}{2} + 3\sin^4\frac{\mu}{2}}{2(4 - 5\sin^2\frac{\mu}{2} + \sin^4\frac{\mu}{2})}\delta^2],$$

$$\beta_y(\delta) = \beta_y[1 - \delta + \frac{3(2 - 3\sin^2\frac{\mu}{2} - \sin^3\frac{\mu}{2} + \sin^4\frac{\mu}{2})}{2(4 - 5\sin^2\frac{\mu}{2} + \sin^4\frac{\mu}{2})}\delta^2].$$

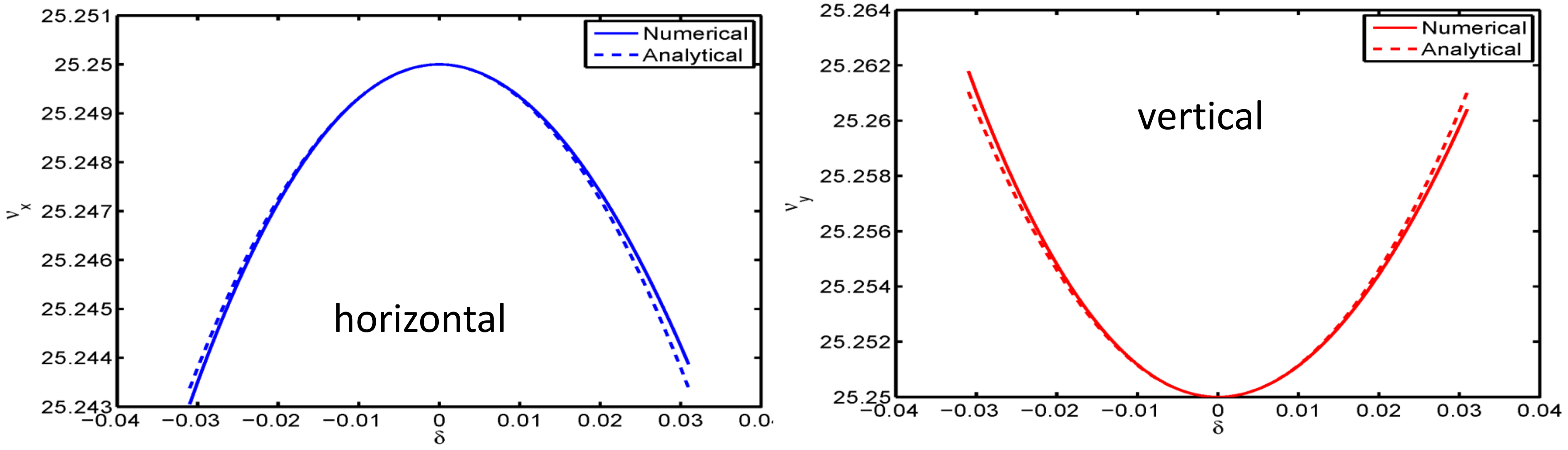


- half integer resonance seen
- not good if $\mu > 135^0$

Second-Order Chromatic Effects

The betatron phase advances up to the second-order of δ:

$$\mu_x(\delta) = \mu - \frac{\tan\frac{\mu}{2}(1-\frac{1}{2}\sin^2\frac{\mu}{2})}{2(1-\frac{1}{4}\sin^2\frac{\mu}{2})}\delta^2, \mu_y(\delta) = \mu + \frac{\tan\frac{\mu}{2}(1+\frac{1}{2}\sin^2\frac{\mu}{2})}{2(1-\frac{1}{4}\sin^2\frac{\mu}{2})}\delta^2.$$



Comparison to a numerical simulation in LEGO in a ring that consists of 101 90^0 cells.

Transfer Map for Thin Quadrupole, Sextupole, Octupole, and Decapole

Transfer map is given by a kick:

$$\mathcal{M}_1 = x,$$

$$\mathcal{M}_2 = p_x - \frac{x}{f} - \frac{\kappa}{2}(x^2 - y^2) - \frac{o}{6}x(x^2 - 3y^2) - \frac{\xi}{24}(x^4 - 6x^2y^2 + y^4),$$

$$\mathcal{M}_3 = y,$$

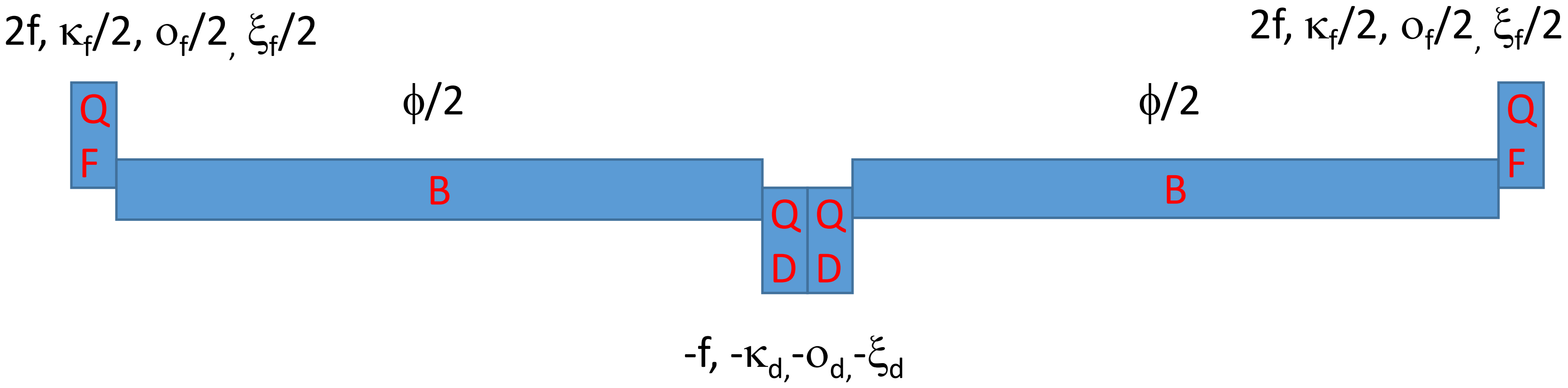
$$\mathcal{M}_4 = p_y + \frac{y}{f} + \kappa xy + \frac{o}{6}y(3x^2 - y^2) + \frac{\xi}{6}xy(x^2 - y^2),$$

$$\mathcal{M}_5 = \delta,$$

$$\mathcal{M}_6 = l,$$

where f is the focusing (in horizontal plane) length of quadrupole and κ, o, ξ are the integrated strengths of sextupole, octupole, and decapole respectively.

Cell with Nonlinear Elements



$$\kappa_f = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1+\frac{1}{2}\sin\frac{\mu}{2})}, \kappa_d = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1-\frac{1}{2}\sin\frac{\mu}{2})},$$

← Sextupoles

$$o_f = \frac{8\sin^5\frac{\mu}{2}}{fL^2\phi^2(1+\frac{1}{2}\sin\frac{\mu}{2})^3}, o_d = -\frac{8\sin^5\frac{\mu}{2}}{fL^2\phi^2(1-\frac{1}{2}\sin\frac{\mu}{2})^3},$$

← Octupoles

$$\xi_f = \frac{16\sin^6\frac{\mu}{2}(4-2\sin\frac{\mu}{2}+\sin^2\frac{\mu}{2})}{fL^3\phi^3(1+\frac{1}{2}\sin\frac{\mu}{2})^5}, \xi_d = \frac{16\sin^6\frac{\mu}{2}(4+2\sin\frac{\mu}{2}+\sin^2\frac{\mu}{2})}{fL^3\phi^3(1-\frac{1}{2}\sin\frac{\mu}{2})^5},$$

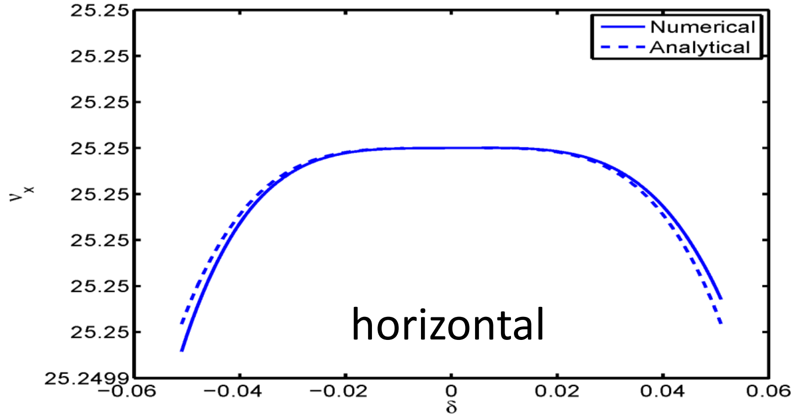
Decapoles

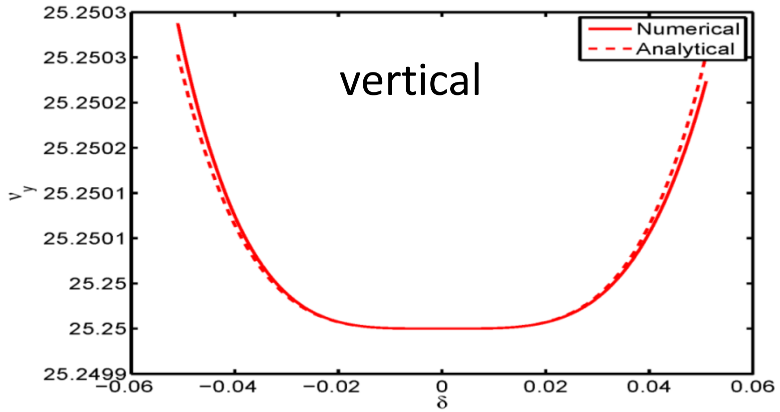
Four-Order Chromatic Effects

The betatron phase advances in the cell up to the fourth-order of δ:

$$\mu_x(\delta) = \mu + \frac{\tan\frac{\mu}{2}(-352 + 312\sin^2\frac{\mu}{2} + 60\sin^4\frac{\mu}{2} + \sin^6\frac{\mu}{2})}{12(4-\sin^2\frac{\mu}{2})^3}\delta^4,$$

$$\mu_y(\delta) = \mu + \frac{\tan\frac{\mu}{2}(992 + 840\sin^2\frac{\mu}{2} + 84\sin^4\frac{\mu}{2} + \sin^6\frac{\mu}{2})}{12(4-\sin^2\frac{\mu}{2})^3}\delta^4.$$





Comparison to a numerical simulation in LEGO in a ring that consists of 101 90^0 cells.

Beta Functions and Phase Advances within cell

The beta functions in region 0<s<L/2, up to the third-order of δ:

$$\beta_x(\delta,s)=\beta_x[1-\frac{4s}{L}\sin\frac{\mu}{2}(1-\frac{2s\sin\frac{\mu}{2}}{L(1+\sin\frac{\mu}{2})})](1-\delta+\delta^2+\delta^3),$$

$$\beta_y(\delta,s)=\beta_y[1+\frac{4s}{L}\sin\frac{\mu}{2}(1+\frac{2s\sin\frac{\mu}{2}}{L(1-\sin\frac{\mu}{2})})](1-\delta+\delta^2+\delta^3).$$

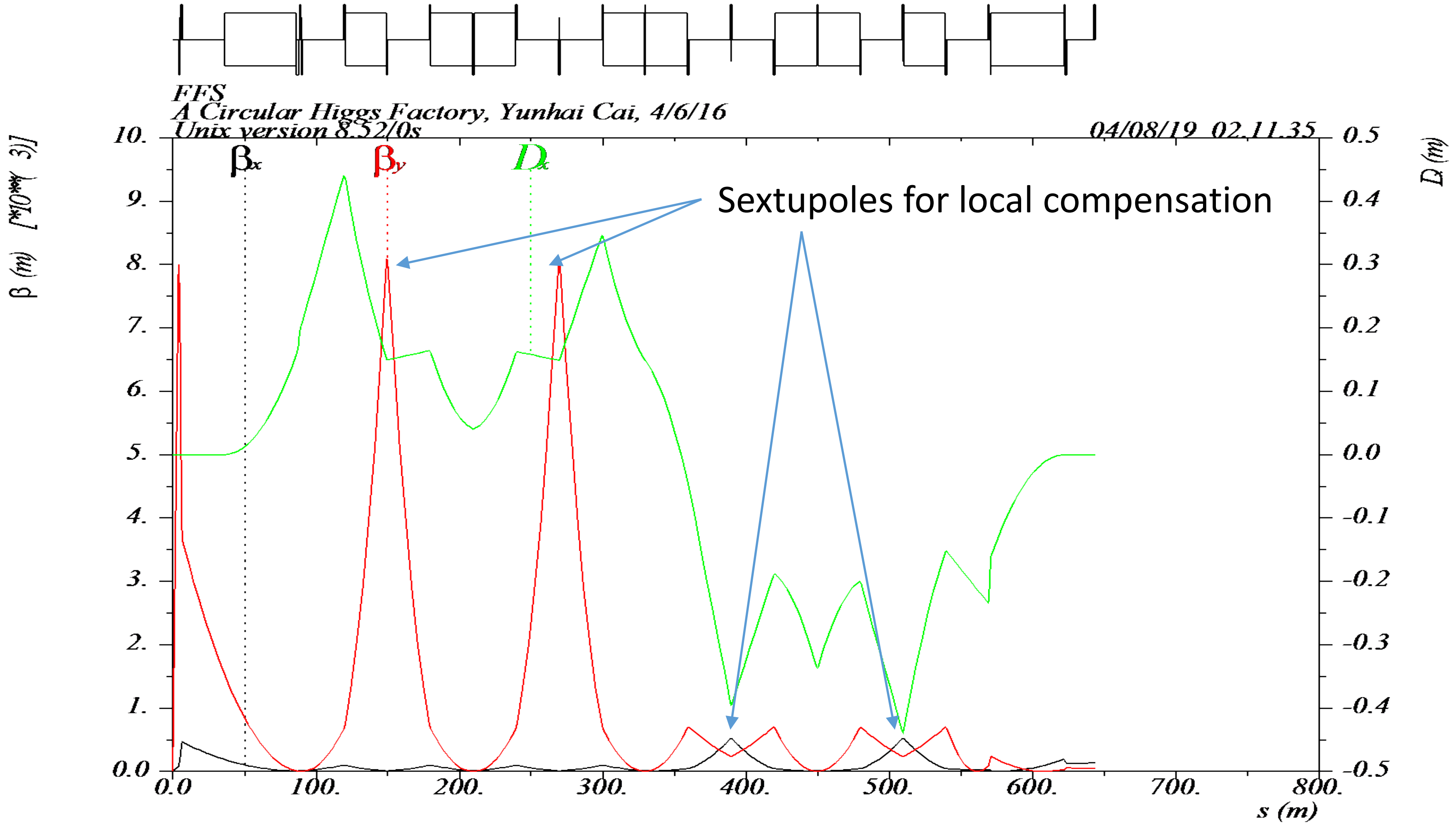
The betatron phase advances up to the third-order of δ:

$$\psi_x(\delta,s)=\tan^{-1}[\frac{\tan\frac{\mu}{2}(1-\sin\frac{\mu}{2})\frac{2s}{L}}{1-\frac{2s}{L}\sin\frac{\mu}{2}}],$$

$$\psi_y(\delta,s)=\tan^{-1}[\frac{\tan\frac{\mu}{2}(1+\sin\frac{\mu}{2})\frac{2s}{L}}{1+\frac{2s}{L}\sin\frac{\mu}{2}}].$$

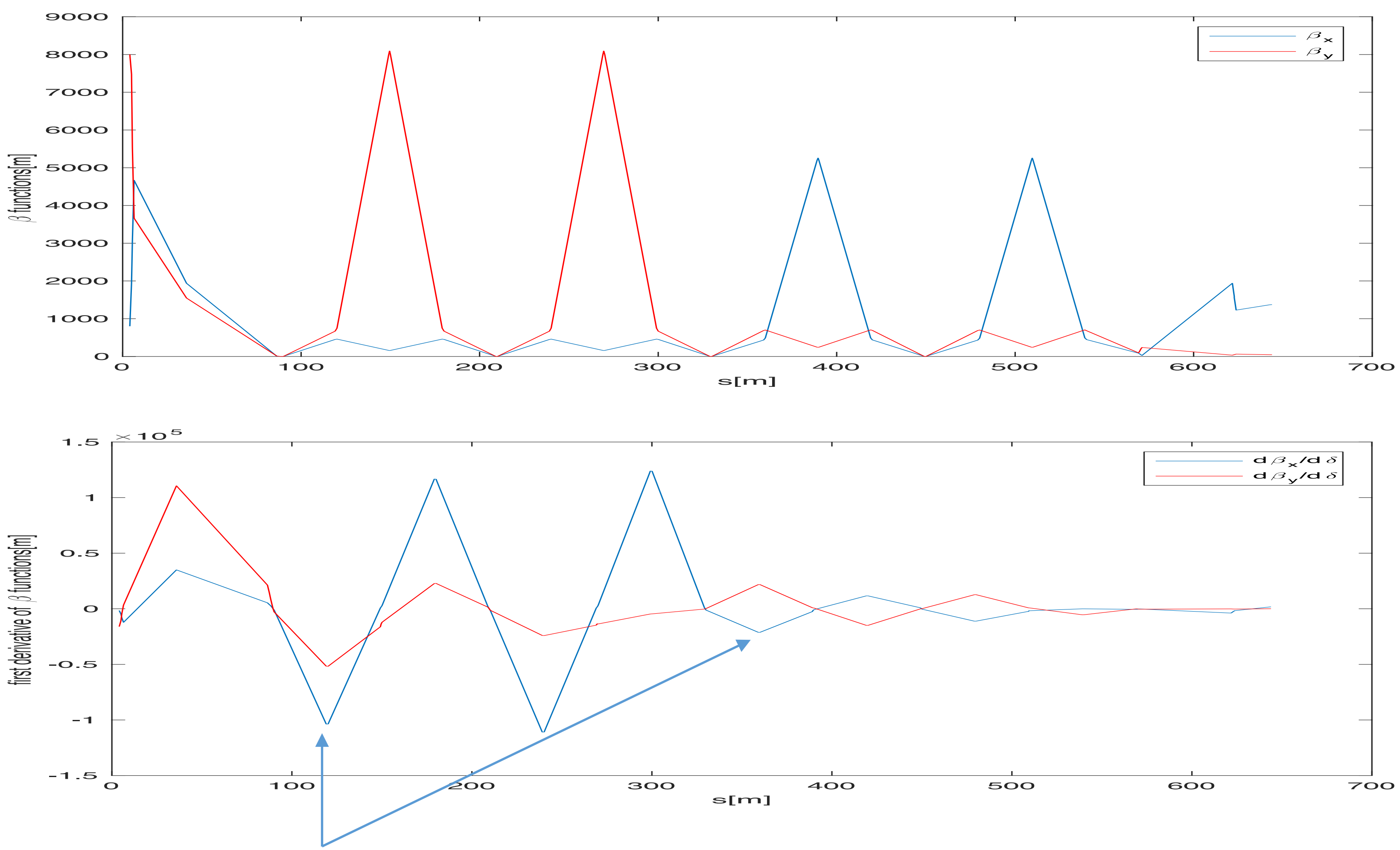
← no δ dependence !

Final Focusing System of a Circular Higgs Factory



β_y^*=0.002 m, β_x^*=0.02 m, L*=4 m

Compensation of Higher Order Chromatic Optics in FFS



1) Brinkmann sextupoles for third-order chromatic phase advances. 2) Octupoles for fourth-order ones.

Differential Algebra

Analytic

Given a function,

$$f(x)=\frac{1}{x+\frac{1}{x}}$$

We know that its derivative

$$f'(x)=-\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2}$$

In particular, for x=2, we have

$$f(2)=\frac{2}{5}$$

$$f'(2)=-\frac{3}{25}$$

TPSA

Rules:

$$(a_0,a_1)+(b_0,b_1)=(a_0+b_0,a_1+b_1)$$

$$\frac{1}{(a_0,a_1)}=(\frac{1}{a_0},-\frac{a_1}{a_0^2})$$

Compute:

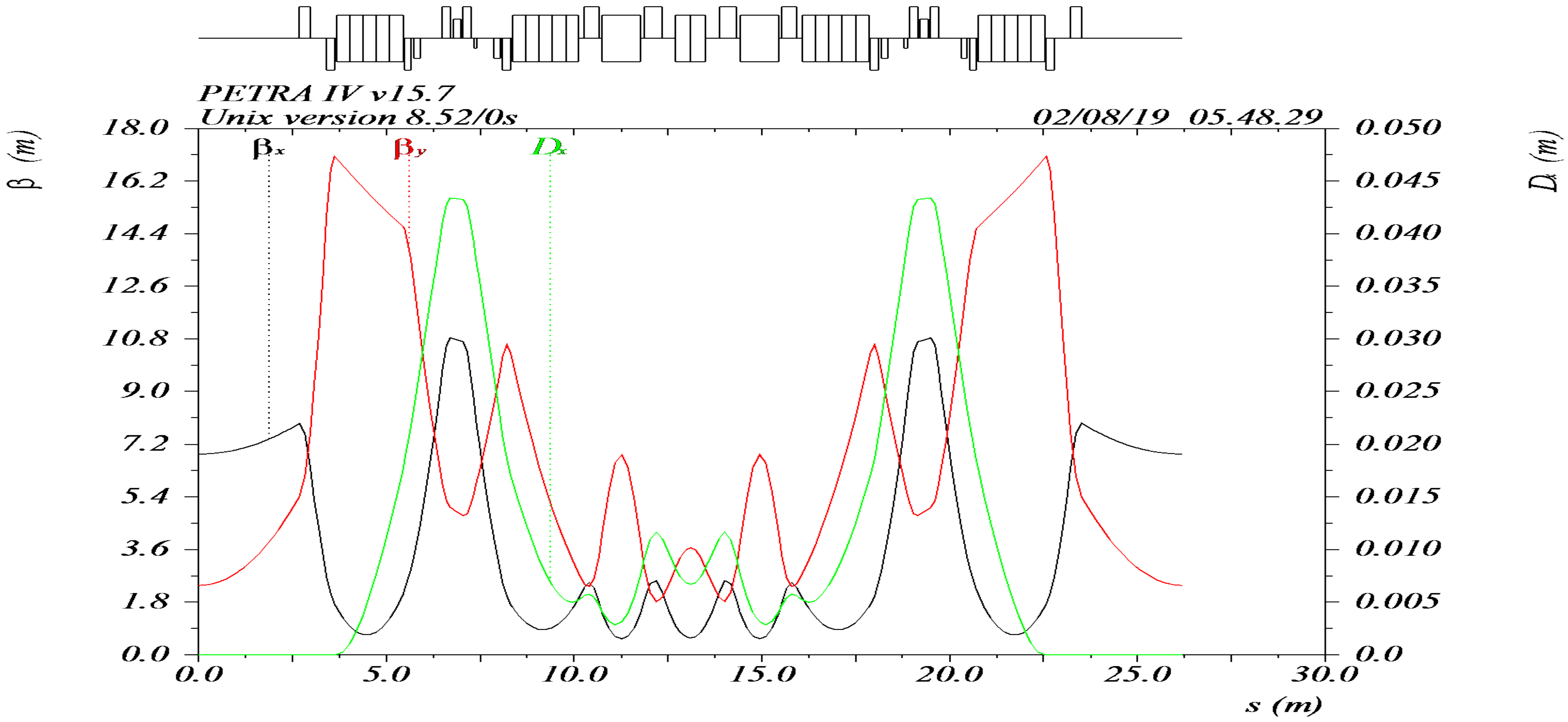
$$\frac{1}{(2,1)+\frac{1}{(2,1)}}=\frac{1}{(2,1)+(\frac{1}{2},-\frac{1}{4})}$$

$$=\frac{1}{(\frac{5}{2},\frac{3}{4})}=(\frac{2}{5},-\frac{3}{25})$$

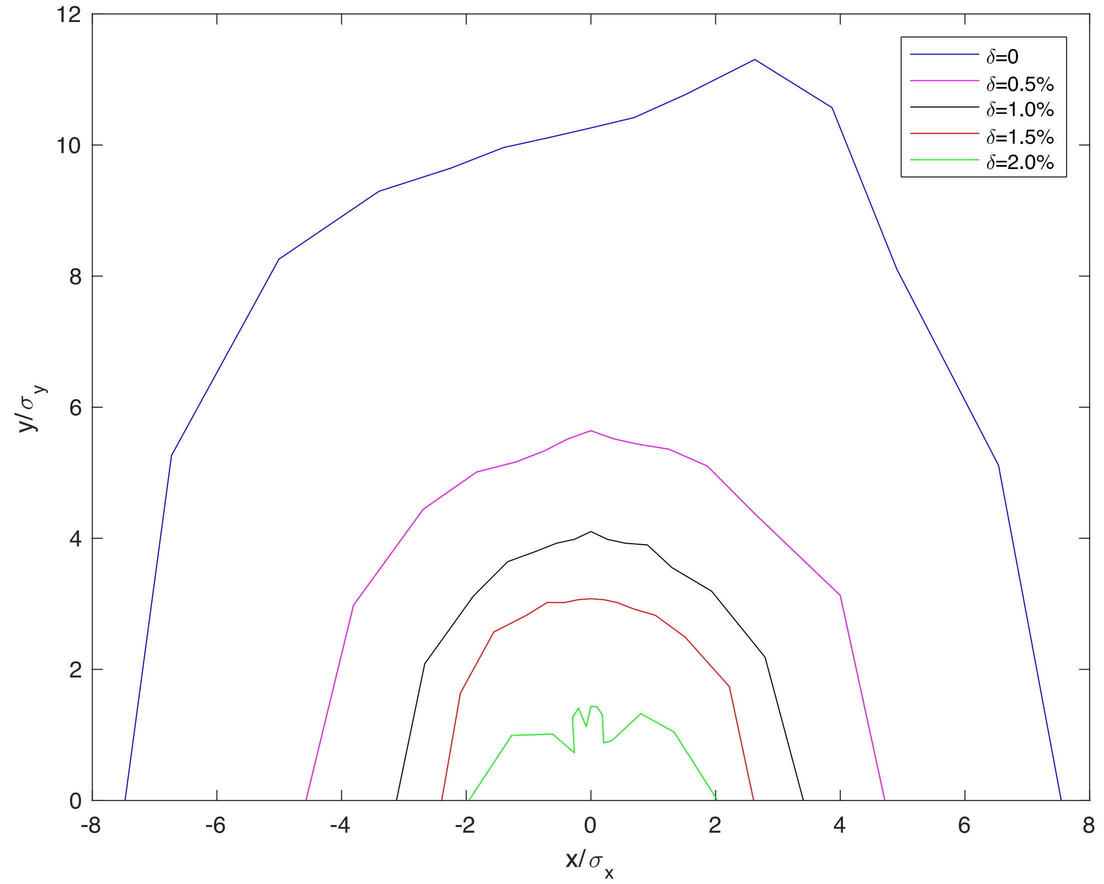
Result in: $f(v)=(f(a_0),f'(a_0))$

Starting: $v=(a_0,1)$

MBA Cell of PETRA IV

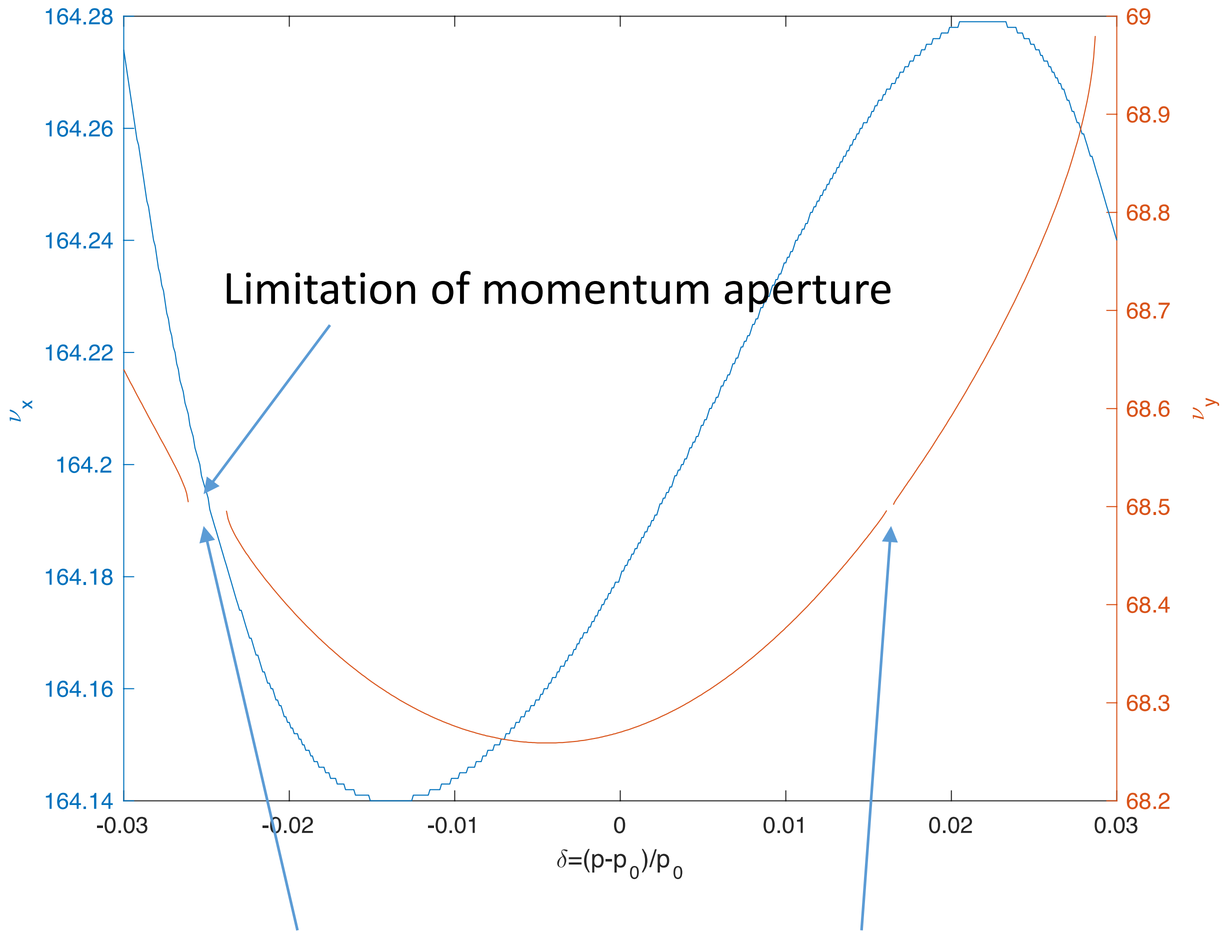


Dynamic Aperture of V15.7



1) Moment aperture is 2%. 2) Synchrotron oscillation is on. 3)Injection emittance: 19 nm.

Chromatic Betatron Tunes of V15.7



Is it safe to cross the half integer resonance? Yes or No

Chromatic Optics of V15.7

	β[m]	α	ν	η[m]	η'
order: 0					
	2.166646e+01	3.763094e-08	1.641800e+02	2.901876e-15	-2.219531e-16
	3.691501e+00	-1.317824e-07	6.827000e+01	0.000000e+00	0.000000e+00
order: 1					
	-6.299379e+01	6.191145e+00	5.009030e+00	-5.975198e-08	-2.044546e-09
	-6.468853e+00	-4.042345e+00	5.010832e+00	0.000000e+00	0.000000e+00
order: 2					
	3.092426e+03	-8.370666e+02	1.079988e+02	1.181970e+01	4.024854e-01
	-3.950714e+03	-4.777085e+02	5.711957e+02	5.087100e-14	3.671072e-14
order: 3					
	-3.373562e+05	1.864965e+04	-3.813990e+03	-4.763566e+02	-1.935361e+01
	5.228626e+04	-4.638407e+03	1.831766e+02	3.376327e-12	-3.738076e-12

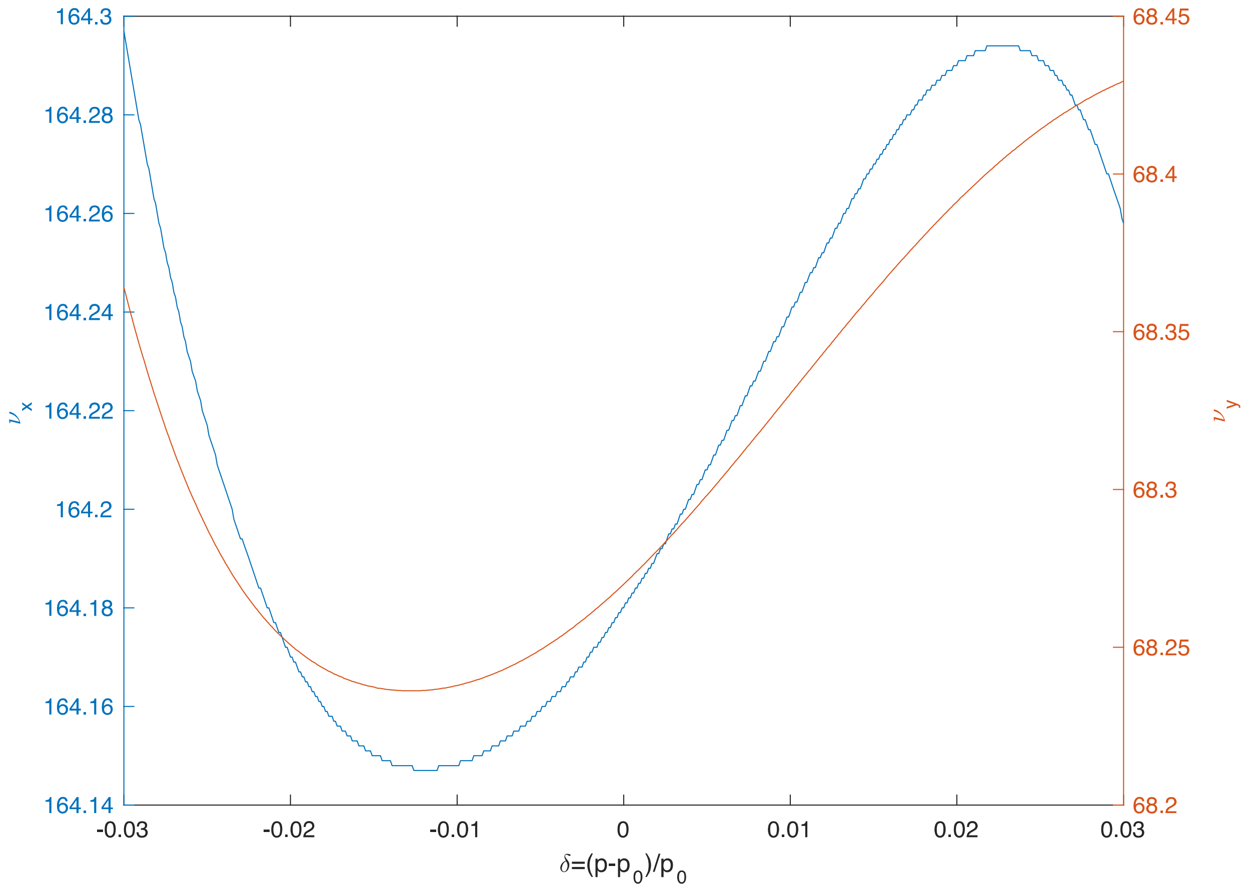
- Differential algebra
- Symplectic maps
- Accurate derivatives
- Arbitrary order
- Include coupling
- Written in C++

Sextupole and Octupole Families and Their Strengths

Names	V15.7		Octupole Solution	
	$K_2[m^{-3}]$	$K_3[m^{-4}]$	$K_2[m^{-3}]$	$K_3[m^{-4}]$
SD1A	-427.39		-424.50	-136222.56
SF2A	378.66		378.74	-30569.90
OF1B		-85229.93		
SD1B	-367.77		-371.70	271306.62
SD1D	-367.77		-371.70	271306.62
OF1D		-85229.93		
SF2E	378.66		378.74	-30569.90
SD1E	-427.39		-424.50	-136222.56

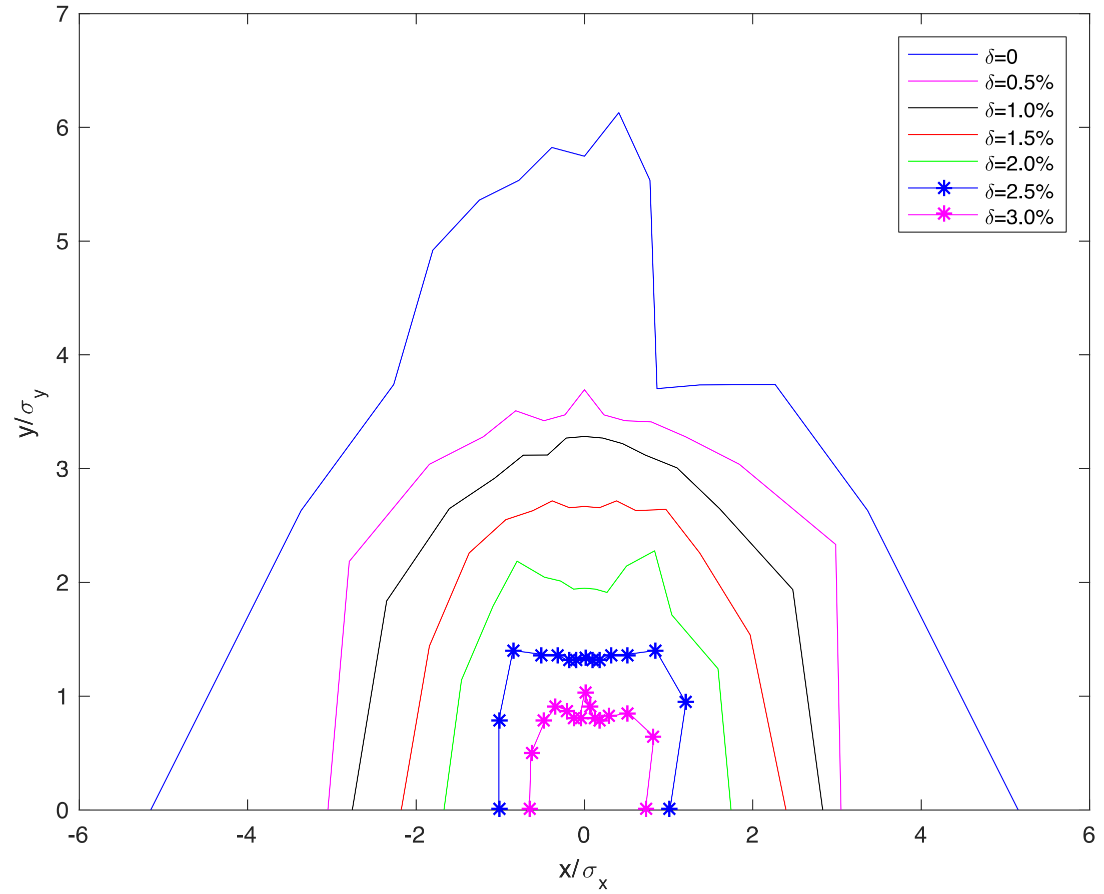
1) Reflection symmetry is retained. 2) All cells are identical.

Chromatic Betatron Tune of Octupole Solution



Solution of chromaticity: ξ_1=5, ξ_2=150, ξ_3=-4000, in both planes

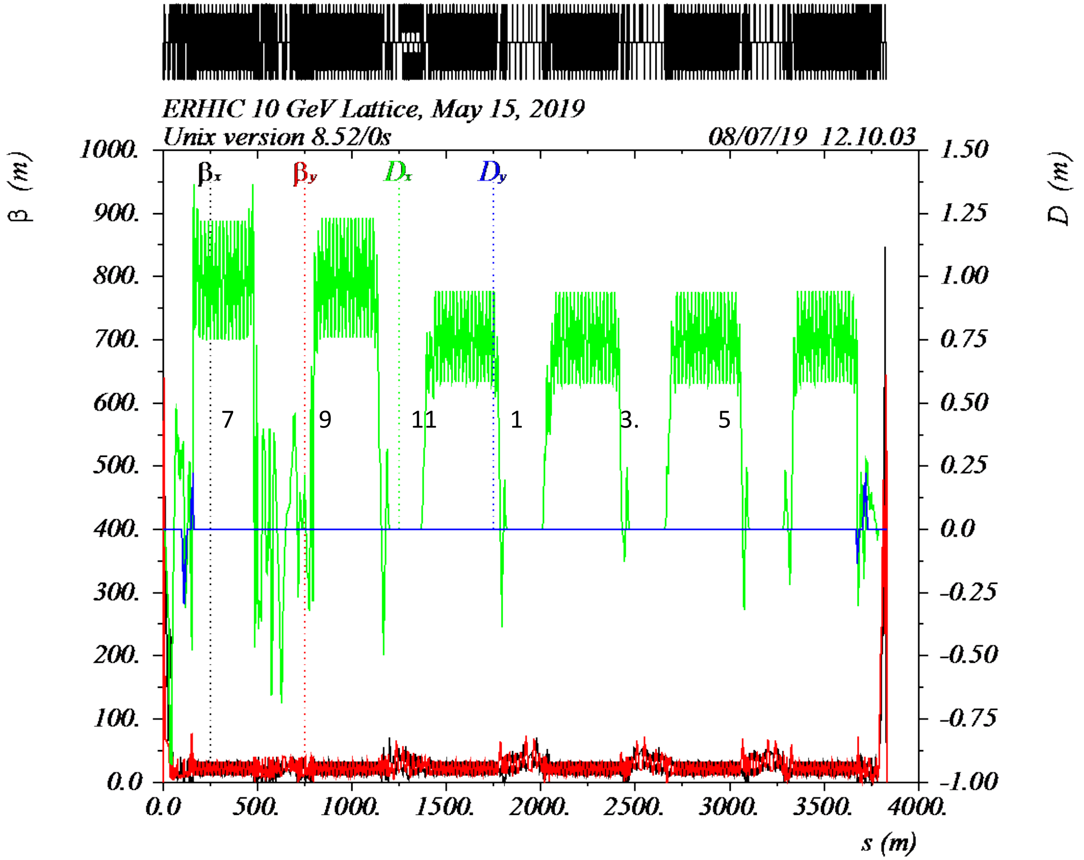
Dynamic Aperture of Octupole Solution



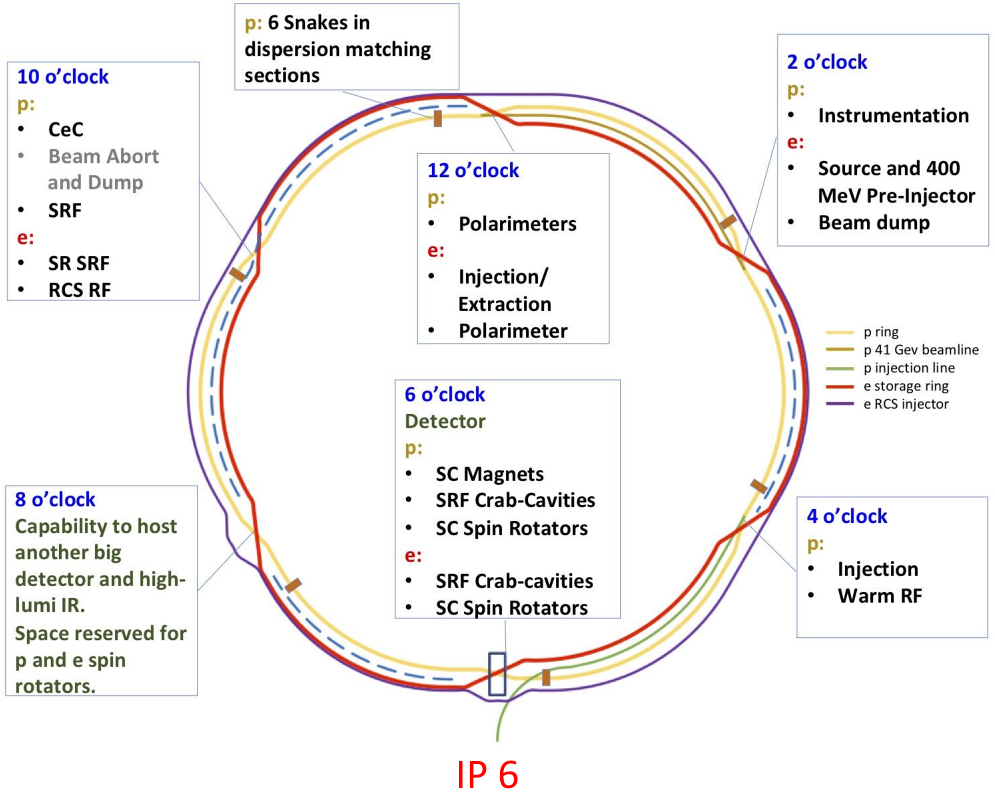
1) Moment aperture increases to 3%. 2) No crossing of half integer resonance. 3) But dynamic aperture is smaller.

eRHIC

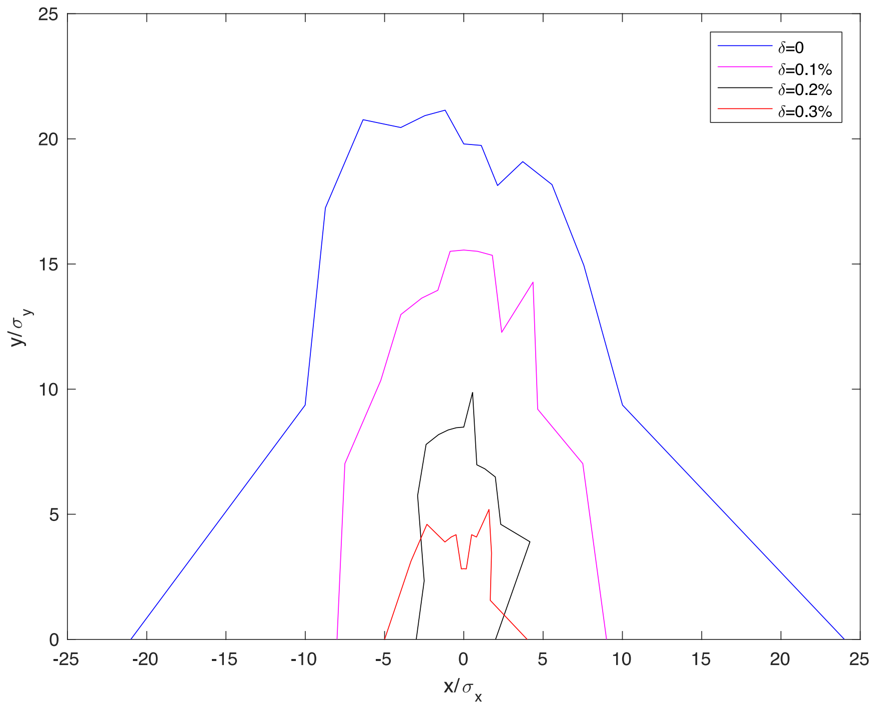
Version 5.0, Tepikian

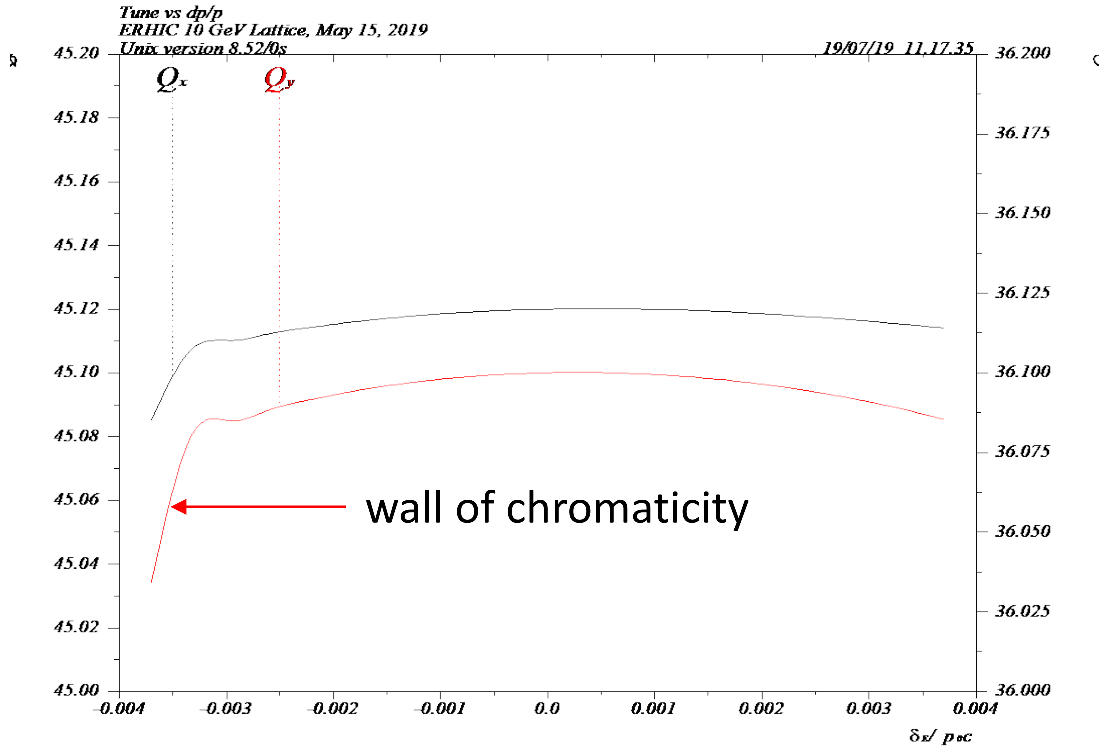


Layout



Dynamic Aperture of eRHIC Lattice

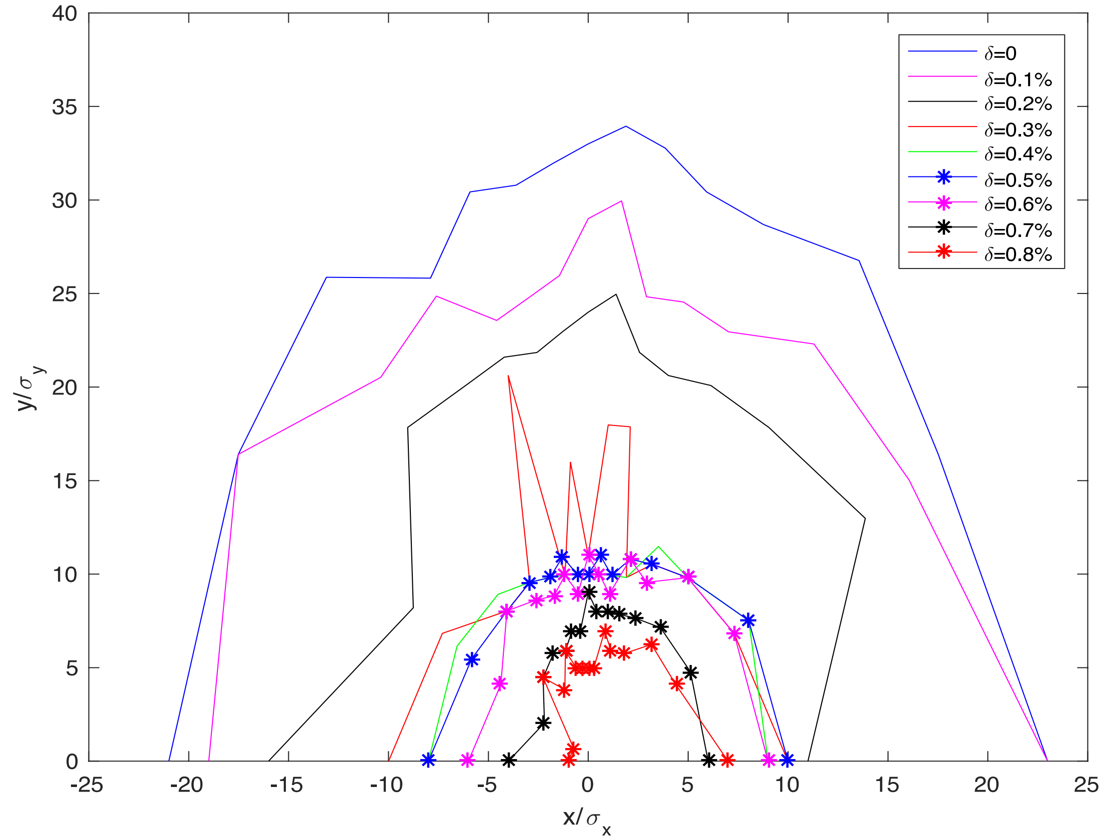




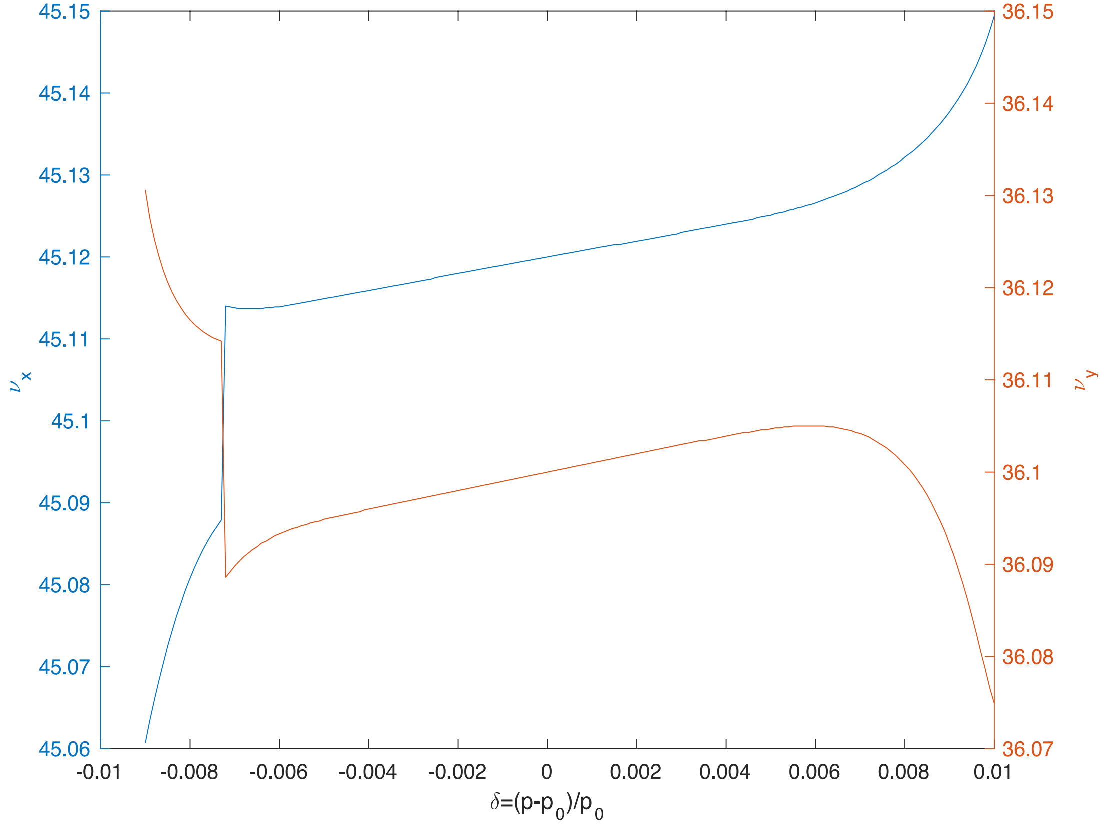
- Reference emittances
  - horizontal 24 nm
  - vertical 12 nm
- Synchrotron oscillation: on

Momentum aperture is 0.3% consistent with momentum bandwidth

Improved Momentum Aperture of eRHIC Lattice

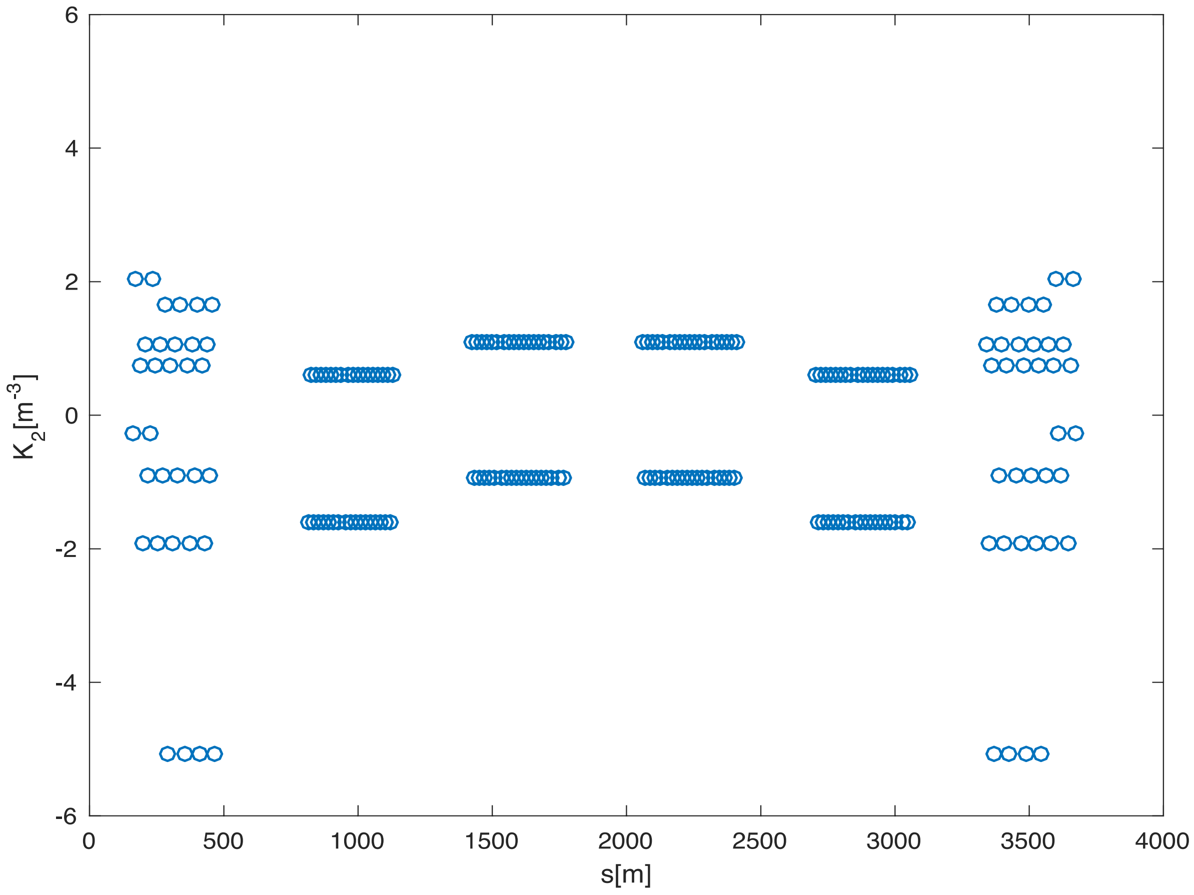


Momentum aperture increases from 0.3% to 0.8%



Chromatic coupling resonance is seen

Strengths of Families of Sextupoles in ERHIC



Chromaticity up to sixth-order of δ is well compensated.

Summary

- We have developed an efficient and accurate method of compute chromatic optics using symplectic maps along with conventional optimizers
- We found that octupoles can be very useful to compensate chromaticity when there is no places for additional sextupoles
- PETRA IV lattice:
  - Crossing half integer resonance can be dangerous
  - Strictive momentum aperture of V15.7 lattice should be 1.5%
  - Momentum aperture can be increased to 3% with more octupole families but with a smaller dynamic aperture
  - A smaller dynamic aperture could be acceptable if beta function at injection point is increased and injecting emittance is reduced
  - Pole tip radius may have to be reduced from 13 mm to 10 mm to accommodate the stronger octupoles, which can be combined to quadrupoles

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