Generalised Ripken Function

Etienne Forest

KEK

N.B. From now on “6” means energy; not 5 as in PTC. Sorry!

Topics

- Generalised Lattice functions
- Viewed as dispersions (stroboscopic average looking at every $1/\nu_s$)
- De Moivre representation in N-degrees of freedom: N representations of the complex numbers.
- Invariants, Moments averages
- Connection between Sands and Chao theory of radiation
- Approximate formulae for the crab crossing angle of Ehrlichman et al.

Topics not covered

- Kinematics invariant in 2 degrees of freedom i.e., extensions of $\langle x^2\rangle\langle p^2\rangle-\langle xp\rangle^2$
- Connection to the Teng-Edwards and the extended Ohmi-Hirata-Oide extension: see section on crab crossing for a taste.
- Same topics: it is possible to re-express simple expression involving only standard Ripken functions in terms of the generalised dispersions and the uncoupled Ripken function. See crab section for a taste. The ζ function of the Japanese is just a generalised dispersion of Ripken.

What I wrote

To the memory of the late Dr. Gerhard Ripken...

1. What is this paper all about?

This main purpose of this paper is pedagogical. I have often heard, from accelerator physicists who possess a far greater knowledge than me, that Sands' theory of radiation[1] is an exact linear theory once transverse coupling is taken into account. This is actually false. In practice the theory proposed by Chao [2] is more exact and can handle full synchro-betatron coupling. Actually, the theory of Chao is itself a small damping approximation of the so-called beam envelope theory.

It is also commonly heard that many features of the Courant-Snyder theory have no equivalents in higher dimension; for example the famous de Moivre formula for the one-turn matrix, Eq. (3.2), has no multidimensional equivalents. In this paper I show that it does have an equivalent and that it is intimately related to the connection between Chao and Sands theory of the so-called H-functions.

It should be said that the fact that the beam envelope theory reduces to Chao's theory and that Chao's theory reduces to Sand's theory is never in doubt. It does not require any hard proof. The reduction of the 21 moments of the beam envelope theory follows from a decomposition of the 21 moments into the eigenphasors of the deterministic linear map. For small damping, away

from linear resonances, only three moments remain; they are the fluctuations of the invariants— the so-called H-functions associated to the tunes 1,2, and 3.

The reduction of Chao into Sands is even clearer. Chao's theory assumes small damping. Sands' theory adds a weak syncho-betatron coupling, i.e., a small "longitudinal" tune ν_3. Therefore it follows that Chao *must reduce* to Sands for vanishingly small ν_3. The hierarchical nature of the assumption demands this. *"Quod erat demonstrandum"* most of you might say very appropriately: no ugly algebra, no equations, just a few words suffice.

Nevertheless papers have been published to link algebraically the theory of Chao to that of Sands— in other words through the manipulation of formulas. Interest in such linkage is not completely ridiculous since Sands theory and its usage, particularly in a design stage, is heavily formulaic: synchrotron integrals with various names such as I_2, I_5, etc... that are simple because they use concepts of a cavity-less ring, namely the dispersions and the standard "Twiss lattice" functions.

The first such linkage is due to Ohmi, Hirata, and Oide. They proved the link algebraically by extending the parametrization of Teng-Edwards to the three dimensional oscillator. Their parametrization allow for the appropriate limit to be taken from formulas of the Chao theory. It was really a constructive proof more concerned with Teng-Edwards than anything else. It is worth noticing that the "flagship" code of KEK authored primarily by Oide, SAD[6], uses the beam envelope theory. It makes a lot of sense in a numerical setting.

The next algebraic linkage by Forest, found in reference [7], uses an extension of the Ripken lattice functions to the non-symplectic theory. This is the approach we use here.

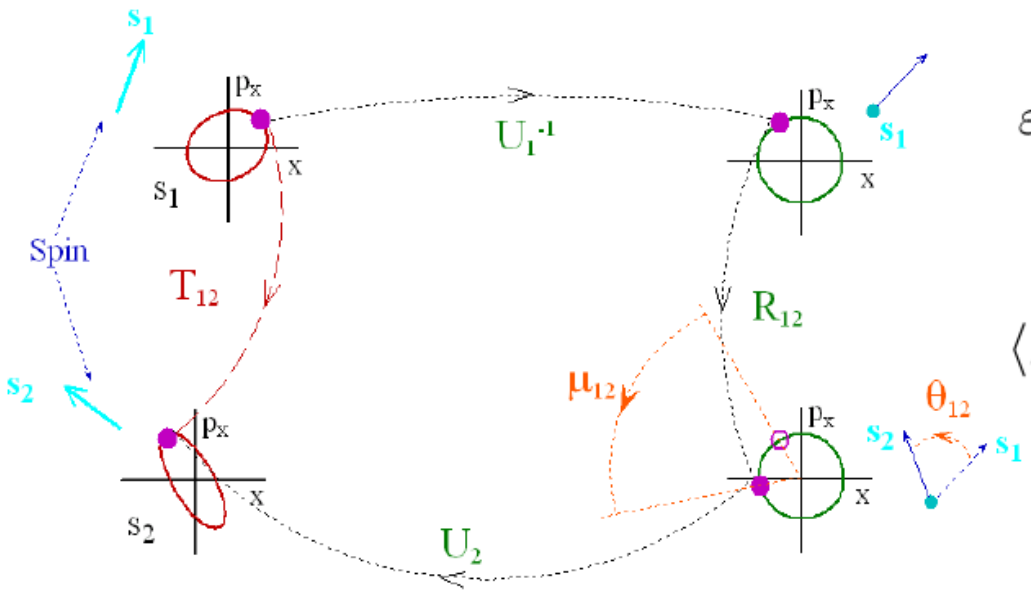
Our only additional insight is the extension of de Moivre formula using the lattice functions of Ripken and Forest. We also slightly modify the dispersion of Forest so as to express it in terms of the average value of the energy deviation δ since δ itself is not a constant in a three dimensional oscillator.

Our final result is simple: if we express the Sands' H-function of the transverse plane in terms of the six dimensional Ripken lattice functions, and then substitute our dispersion, which is also made of six dimensional Ripken-like functions, we obtain a result which agrees **exactly** with the single lattice function of Chao's theory.

As we said before, we should not have expected an exact agreement but simply agreement in the limit of small synchro-betatron coupling. We refer to this result as "amusing" because, in the absence of deep understanding, one must consider this result to be no more than an amusing fluke.

Is this result of immediate practical importance? No. I am the first one to admit that the results of this paper are more amusing than useful. But I am also convinced that the general ignorance of Chao's methods, not to mentioned the beam envelope, is certainly not amusing. Automatic differentiation methods render approximate formulas, like Sands' integrals, inappropriate for general computer implementation and *actually more complex to implement* than the more correct linear theories.

Standard Ripken Lattice Function



$$\varepsilon_{p(a)} = \left\{ A^{-1}_{ai} A^{-1}_{aj} + A^{-1}_{\bar{a}i} A^{-1}_{\bar{a}j} \right\} z_i z_j$$

$$\langle z_i z_j \rangle = \frac{1}{2} \left\{ A_{ia} A_{ja} + A_{i\bar{a}} A_{j\bar{a}} \right\} \varepsilon_{p(a)}$$

$$(-1)^{i+a} A_{ia} = A^{-1}_{\bar{a}\bar{i}}$$

Notation: $\bar{a} = a + (-1)^{a+1}$

$$(-1)^{i+j} \left\{ A^{-1}_{\bar{a}\bar{i}} A^{-1}_{\bar{a}\bar{j}} + A^{-1}_{a\bar{i}} A^{-1}_{a\bar{j}} \right\}$$

One implication: if you measure the tune of a particle and the average moments
You can reconstruct the matrix!!!!

Invariant in 1-d-f

$$\underbrace{\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}}_{\text{Courant-Snyder Choice}} = \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{rotation of } -\pi/2}$$

$$A = \begin{pmatrix} 1/\sqrt{\gamma} & -\alpha/\sqrt{\gamma} \\ 0 & \sqrt{\gamma} \end{pmatrix}$$

$$\begin{aligned}
\varepsilon &= z_1^{new2} + z_2^{new2} \\
&= \left(A_{11}^{-1} z_1 + A_{12}^{-1} z_2\right)^2 + \left(A_{21}^{-1} z_1 + A_{22}^{-1} z_2\right)^2 \\
&= \left\{ \begin{array}{c} \underbrace{\left(\sqrt{\beta} z_1\right)^2 + \left(\frac{\alpha}{\sqrt{\beta}} z_1 + \frac{1}{\sqrt{\beta}} z_2\right)^2}_{\text{Courant-Snyder Choice}} \\ \text{or} \\ \underbrace{\left(-\frac{\alpha}{\sqrt{\beta}} z_1 - \sqrt{\beta} z_2\right)^2 + \left(\frac{1}{\sqrt{\beta}} z_1\right)^2}_{\text{Choice of } Eq.\ (2.44)} \end{array} \right\} = \underbrace{\frac{1+\alpha^2}{\beta}}_{\gamma \text{ per } Eq.\ (2.29)} z_1^2 + 2\alpha z_1 z_2 + \beta z_2^2 \\
&= \gamma z_1^2 + 2\alpha z_1 z_2 + \beta z_2^2
\end{aligned} \tag{2.47}$$

Averages 1-d-f

$$\begin{aligned}\left\langle z_1^2\right\rangle &= \left\langle (A_{11}z_1^{new}+A_{12}z_2^{new})^2\right\rangle = A_{11}^2\left\langle z_1^{new2}\right\rangle + A_{12}^2\left\langle z_2^{new2}\right\rangle + 2A_{11}A_{12}\left\langle z_1^{new}z_2^{new}\right\rangle \\ &= \underbrace{A_{11}^2+A_{12}^2}_{\beta}\frac{\left\langle r^2\right\rangle}{2} = \underbrace{A_{11}^2+A_{12}^2}_{\beta}\frac{r^2}{2} \end{aligned} \tag{2.50}$$

$$\begin{aligned}\left\langle z_2^2\right\rangle &= \left\langle (A_{21}z_1^{new}+A_{22}z_2^{new})^2\right\rangle = A_{21}^2\left\langle z_1^{new2}\right\rangle + A_{22}^2\left\langle z_2^{new2}\right\rangle + 2A_{21}A_{22}\left\langle z_1^{new}z_2^{new}\right\rangle \\ &= \underbrace{A_{21}^2+A_{22}^2}_{\gamma}\frac{\left\langle r^2\right\rangle}{2} = \underbrace{A_{21}^2+A_{22}^2}_{\gamma}\frac{r^2}{2} \end{aligned} \tag{2.51}$$

$$\begin{aligned}\left\langle z_1z_2\right\rangle &= \left\langle (A_{11}z_1^{new}+A_{12}z_2^{new})\,(A_{21}z_1^{new}+A_{22}z_2^{new})\right\rangle \\ &= A_{11}A_{21}\left\langle z_1^{new2}\right\rangle + A_{12}A_{22}\left\langle z_2^{new2}\right\rangle + \{A_{11}A_{22}+A_{12}A_{21}\}\left\langle z_1^{new}z_2^{new}\right\rangle \\ &= \underbrace{A_{11}A_{21}+A_{12}A_{22}}_{-\alpha}\frac{\left\langle r^2\right\rangle}{2} = \underbrace{A_{11}A_{21}+A_{12}A_{22}}_{-\alpha}\frac{r^2}{2} \end{aligned} \tag{2.52}$$

Generalised Ripken?

WHY?

It all started with dispersion…

$$M \begin{pmatrix} f \\ 0 \\ \delta \end{pmatrix} = \underbrace{\begin{pmatrix} N & 0 & v \\ w & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}}_{6\times 6 \text{ matrix}} \begin{pmatrix} f \\ 0 \\ \delta \end{pmatrix} = \begin{pmatrix} f \\ \tau \\ \delta \end{pmatrix}$$

$$Nf + \delta v = f \;\Rightarrow\; f = \delta(1-N)^{-1}v \;\Longrightarrow \eta = (1-N)^{-1}v$$

$$\vec{f} = \vec{\eta}\,\delta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \delta$$

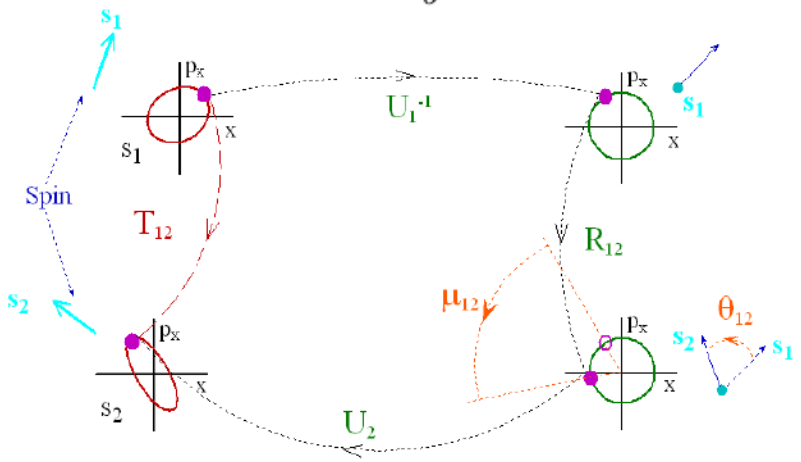
No cavity

1. Accelerators physicists use an ultimately irrelevant concept while designing their machines.
2. The concept of dispersion retains in some adiabatic sense its meaning when the cavities are turned on, that is to say, when time and energy start undergoing harmonic oscillations.

$$\begin{aligned}
\eta_a = \frac{d}{dz_6}\langle z_a \rangle_{1,2} &= \left\{ A_{a6} A_{66}^{-1} + A_{a5} A_{56}^{-1} \right\} \\
&= \left\{ A_{a6} A_{55} - A_{a5} A_{56} \right\} \\
&= (-1)^a \left\{ A_{5\overline{a}}^{-1} A_{66}^{-1} - A_{6\overline{a}}^{-1} A_{56}^{-1} \right\} \\
\text{where } \overline{a} &= a + (-1)^{a+1}
\end{aligned}$$

Derivation done circa 1990

$$w_a = \sum_b A^{-1}_{ab} z_b = A^{-1}_{ab} z_b \quad \leftarrow \text{ Einstein summation convention}$$

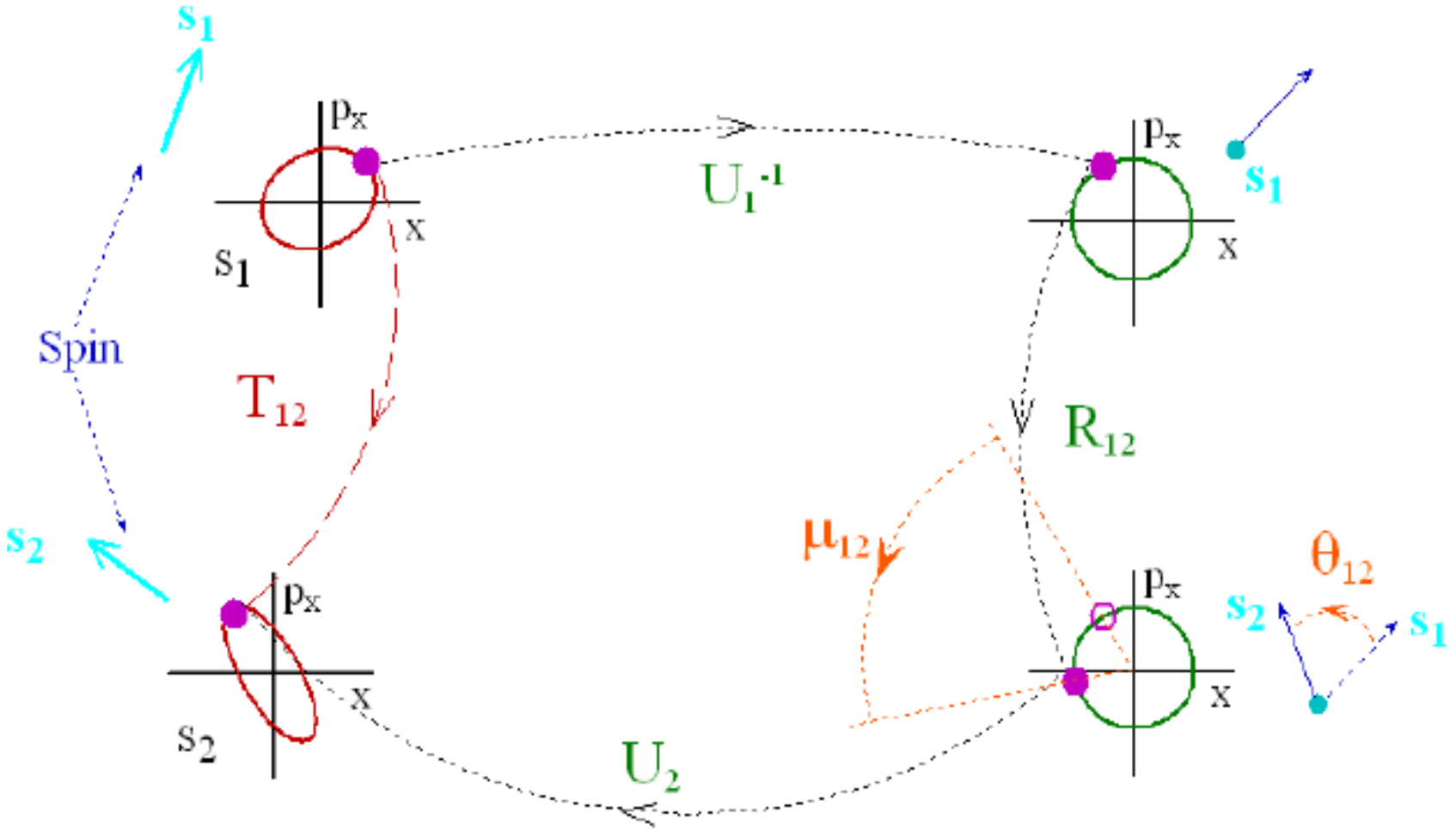


$$\langle \vec{w} \rangle_{1,2} = \left(0, 0, 0, 0, A^{-1}_{5b} z_b, A^{-1}_{6b} z_b\right)$$

$$\vec{H} = \langle \vec{z} \rangle_{1,2} = A \langle \vec{w} \rangle_{1,2} \Longrightarrow H_a = \left\{ A_{a5} A^{-1}_{5b} + A_{a6} A^{-1}_{6b} \right\} z_b$$

Coefficient of z_6

$$\eta_a = \left\{ A_{a6} A^{-1}_{66} + A_{a5} A^{-1}_{56} \right\} = \left\{ A_{a6} A_{55} - A_{a5} A_{56} \right\}$$


s1
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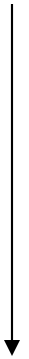
s2
U2

Compare Ripken and dispersion

$$\varepsilon_{p(a)} = \left\{ A_{ai}^{-1} A_{aj}^{-1} + A_{\overline{a}i}^{-1} A_{\overline{a}j}^{-1} \right\} z_i z_j$$

$$\langle z_i z_j \rangle = \frac{1}{2} \left\{ A_{ia} A_{ja} + A_{i\overline{a}} A_{j\overline{a}} \right\} \varepsilon_{p(a)}$$



?

$$\eta_a = \left\{ A_{a6} A_{66}^{-1} + A_{a5} A_{56}^{-1} \right\} = \left\{ A_{a6} A_{55} - A_{a5} A_{56} \right\}$$

General Theory: 1-d-f

$$M = \cos(\mu) H + \sin(\mu) B$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \text{ with } \mathrm{B}^2 = -\mathrm{H}$$

$$\mathscr{H} = \frac{\mu}{2} z^{\mathrm{T}} K z$$

$$M = \exp(\mu B) = \exp(\mu S K) \longrightarrow M^N = \cos(N\mu) H + \sin(N\mu) B$$

$$K = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \Longrightarrow \varepsilon(z) = z^{\mathrm{T}} K z = \gamma z_1^2 + 2\alpha z_1 z_2 + \beta z_2^2$$

$$B = SK$$

$$B = ES$$

$$K = SES^{\mathrm{T}}$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$E = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \Longrightarrow \langle z_i z_j \rangle = E_{ij} \left\langle \frac{z_1^2 + z_2^2}{2} \right\rangle = E_{ij} \langle J \rangle$$

for example $\langle z_1^2 \rangle = \beta \langle J \rangle$

N-d-f

$$M = A\Lambda R A^{-1}$$

$$R = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \qquad r_i = \begin{pmatrix} \cos\,\mu_i & \sin\,\mu_i \\ -\sin\,\mu_i & \cos\,\mu_i \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \qquad \Lambda_i = \begin{pmatrix} \exp(-\alpha_i) & 0 \\ 0 & \exp(-\alpha_i) \end{pmatrix}$$

$$M = A\left(\sum_i e^{-\alpha_i}\left\{\cos(\mu_i)\, I^i + \sin(\mu_i)\, SI^i\right\}\right)A^{-1}$$

$$= \sum_i e^{-\alpha_i}\left\{\cos(\mu_i)\underbrace{AI^iA^{-1}}_{H^i} + \sin(\mu_i)\underbrace{ASI^iA^{-1}}_{B^i}\right\}$$

$$I^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{aligned}
H^i H^j &= \delta_{ij} H^j \\
H^i B^j &= B^j H^i = \delta_{ij} B^j \\
B^i B^j &= -\delta_{ij} H^j
\end{aligned}
$$

De Moivre's general formula follows from the three previous equations.

$$
M^N = \sum_i e^{-N\alpha_i} \left\{ \cos\left(N\mu_i\right) H^i + \sin\left(N\mu_i\right) B^i \right\}
$$

Lie Operator and Dispersion

$$M = \exp\left(\sum_i -\alpha_i H^i + \mu_i B^i\right) = \exp\left(\sum_i -\alpha_i H^i + \mu_i S K^i\right)$$

$$\mu_3 \approx 0 \ll \mu_1, \mu_2 \quad \Longrightarrow \quad \langle M \rangle_N = \left\langle \sum_{i=1,3} \left\{ \cos\left(N 2\pi \nu_i\right) H^i + \sin\left(N 2\pi \nu_i\right) B^i \right\} \right\rangle_N$$

$$\approx H^3$$

$$\text{if} \qquad 1/\nu_3 \gg N \gg 1/\nu_1, 1/\nu_2$$

$$\vec{f} = H^3 \vec{z} \qquad f_a = H^3_{a6}\, z_6 \quad \text{or} \quad \vec{\mathrm{f}} = \left(H^3_{16}, H^3_{26}, H^3_{36}, H^3_{46}, H^3_{56}, H^3_{66}\right) z_6$$

$$f_a = \eta_a f_6 \;; \quad \text{for a} = 1,4 \quad \text{or} \quad \vec{\eta} = \left(\frac{H^3_{16}}{H^3_{66}}, \frac{H^3_{26}}{H^3_{66}}, \frac{H^3_{36}}{H^3_{66}}, \frac{H^3_{46}}{H^3_{66}}\right)$$

Application : Crab crossing angle

Cornell Mafia

7. Some Comments on the Crab Angle Calculation by M. P. Ehrlichman et al. in reference [5]

In an invariant setting, I believe that the most natural lattice functions are to be defined by the tensors E or K Eq. (5.13). This defintion in any number of degrees of freedom is invariant under the choice of the canonical transformation A of Eq. (4.1).

These lattice functions do not exploit the relative smallness of the longitudinal tune μ_3. In particular, for nearly frozen longitudinal motion, there appears a quasi-fix point $\vec{f}$ given by Eq. (5.8):

$$\vec{f} = H^3 \vec{z} \tag{7.1}$$

It is remarkable that this quasi-fixed point can be obtained trivially (see Eq. (5.6)) from the non-Hamiltonian lattice tensors H^i which enter in the multi-dimensional formula de Moivre formula as well as in non-Hamiltonian perturbation theory (see [4] for example).

The most famous quasi-fixed point is the dispersion. It is given by the z_6 component of $\vec{f}$:$\eta_i = H^3_{i6}$. Less known, due to the accidental nature of accelerator physics, is the dispersion ζ of reference [8]. It is given by $\zeta_i = H^3_{i5}$.

We have seen in this paper that the dispersion η enters in a derivation of the Sands integrals starting from the Chao formulas. Moreover all my formulas are tensorial in nature: they are made of dot products and wedge products of A with its inverse. In the symplectic case, they can be made of products of A with itself or A^{-1} with itself.

The entire theory is independent of a cooked up factorization such as the Teng-Edwards extension of the Japanese in reference [8]. One just tracks the A in a Twiss loop and evaluate the functions trivially by contracting A with itself in the symplectic case.

7.1 The Crab Crossing Angle

According to Eq. (30) of Ehrlichman et al., the crossing angles are given by:

$$\theta_{ij} = \frac{1}{2} \tan\left(\frac{2\sigma_{ij}}{\sigma_{ii} - \sigma_{jj}}\right) \tag{7.2}$$

According to Eqs. (3.3), (3.4) and (3.5) the computation of the σ_{ij} are all connected to the Ripken symplectic functions B^i. I reproduce them here:

$$B = SK \tag{7.3}$$
$$B = ES \tag{7.4}$$
$$K = SES^{\mathrm{T}} \tag{7.5}$$

In terms of the Ripken-lattice function, the answer is trivial:

$$\Theta_{51} = \frac{1}{2}\tan^{-1}(\theta_{51}) = \frac{\sum\limits_{a=1,3} E^a_{51}\varepsilon_a}{\sum\limits_{a=1,3} E^a_{55}\varepsilon_a - E^a_{11}\varepsilon_a} \tag{7.6}$$

In the case when

$$\varepsilon_3 \gg \varepsilon_1 \text{ or } \varepsilon_2 \tag{7.7}$$

then

$$\Theta_{51} = \frac{1}{2}\tan^{-1}(\theta_{51}) = \frac{E^3_{51}}{E^3_{55} - E^3_{11}} \tag{7.8}$$

And finally, for small coupling:

$$\theta_{51} \approx \frac{E^3_{51}}{E^3_{55}} \tag{7.9}$$

All the E^3 can be expressed from B^3 and the K^3. Remembering that the E's and K's are symmetric matrices, we can rewrite Eq. (7.9) as

$$\theta_{51} \approx \frac{K^3_{26}}{K^3_{66}} \tag{7.10}$$

Now, using Eq. (5.11), I can write

$$K^3_{26} = B^3_{16} \tag{7.11}$$

and finally using the fact that H^3 and B^3 are a representation of the complex numbers (de Moivre), I can write (using $B_{16} = \sum\limits_{a=1,6} H_{1a}B_{a6}$):

$$\begin{aligned} B^3_{16} &= \frac{1}{1-H^3_{11}} \sum_{a=2,6} H^3_{1j}B^3_{j6} \\ &= H^3_{15}B^3_{56} + H^3_{16}B^3_{66} + O(2) \text{ in coupling} \end{aligned} \tag{7.12}$$

But, from Eq. (7.1) we have the result:

$$\zeta_1 = H^3_{15} \quad \text{and} \quad \eta_1 = H^3_{16} \tag{7.13}$$

And from the connection between B^3 and K^3:

$$\beta_3 = K^3_{66} = B^3_{56} \quad \text{and} \quad \alpha_3 = K^3_{56} = -B^3_{66} \tag{7.14}$$

We then obtained the Japanese result:

$$\theta_{51} \approx \zeta_1 - \frac{\alpha_3}{\beta_3}\eta_1 \tag{7.15}$$

Of course all my lattice functions (symplectic B^3 and dispersive H^3) are all invariants which **do not** depend on any crazy parametrizations and thus can be directly extracted from a simple propagation of the matrix A. No Teng-Edwards-Ohmi-Hirata-Oide nonsense is necessary.

$$\langle \Delta\varepsilon_i \rangle_{\text{Sands-Like}} = \left\{ \sum_{a,b=1,4} K^i_{ab} \eta_a \eta_b \right\} \frac{\langle \Delta^2 \rangle}{2\alpha_i}$$

$$= \left\{ \frac{1}{\left(H^3_{66}\right)^2} \sum_{a,b=1,4} K^i_{ab} H^3_{a6} H^3_{b6} \right\} \frac{\langle \Delta^2 \rangle}{2\alpha_i}$$

$$\eta_a = \frac{d}{d\langle z_6 \rangle_{1,2}} \langle z_a \rangle_{1,2} = \frac{A_{a6}A^{-1}_{66} + A_{a5}A^{-1}_{56}}{A_{66}A^{-1}_{66} + A_{65}A^{-1}_{56}}$$

$$= \frac{A_{a6}A_{55} - A_{a5}A_{56}}{A_{66}A_{55} - A_{65}A_{56}}$$

$$= (-1)^a \frac{A^{-1}_{5\overline{a}}A^{-1}_{66} - A^{-1}_{6\overline{a}}A^{-1}_{56}}{A^{-1}_{55}A^{-1}_{66} - A^{-1}_{65}A^{-1}_{56}}$$

where $\overline{a} = a + (-1)^{a+1}$

The proof start with a simple equation:

$$\text{for } i \neq 3 \;\rightarrow\; {H^3}^{\mathrm{T}} K^i H^3 = 0 \tag{6.7}$$

We can understand Eq. (6.7) through simple mathematics first. First we can prove that

$$K^i H^3 = 0$$

$$\underbrace{\left(K^i H^3\right)^{\mathrm{T}} = {H^3}^{\mathrm{T}} K^i}_{K^i \text{ is symmetric}} = 0 \tag{6.8}$$

$$H^i H^j = \delta_{ij} H^j$$

$$H^i B^j = B^j H^i = \delta_{ij} B^j$$

$$B^i B^j = -\delta_{ij} H^j$$

since

$$\begin{aligned} K^i H^3 &= S^{\mathrm{T}} S K^i H^3 \\ &\downarrow (5.11) \\ &= S^{\mathrm{T}} B^i H^3 \\ &= S^{\mathrm{T}} \underbrace{B^i H^3}_{=0 \ (4.7)} = 0 \end{aligned} \tag{6.9}$$

the third tune ν_3. Therefore such a ray cannot have any component in the first two planes: Eq. (6.7) must be satistifed.

To proceed further, we denote a summation over the "transverse" indices $1,2,3,4$ by the symbol $\otimes$ and a summation over the "longitudinal" indices $5,6$ by $\times$. Thus the product of two matrices A and B can be written as:

$$AB = A \otimes B + A \times B \tag{6.10}$$

We can rewrite Eq. (6.8)

$$\begin{aligned} K^i \otimes H^3 + K^i \times H^3 &= 0 \\ H^{3\mathrm{T}} \otimes K^i + H^{3\mathrm{T}} \times K^i &= 0 \end{aligned} \tag{6.11}$$

We finally apply Eq. (6.10) to Eq. (6.7):

$$\begin{aligned} H^{3\mathrm{T}} K^i H^3 &= 0 \\ &\Downarrow \\ 0 &= H^{3\mathrm{T}} \otimes K^i H^3 + H^{3\mathrm{T}} \times K^i H^3 \\ 0 &= H^{3\mathrm{T}} \otimes \left\{ K^i \otimes H^3 + K^i \times H^3 \right\} + H^{3\mathrm{T}} \times \left\{ K^i \otimes H^3 + K^i \times H^3 \right\} \\ 0 &= H^{3\mathrm{T}} \otimes K^i \otimes H^3 + H^{3\mathrm{T}} \otimes K^i \times H^3 + H^{3\mathrm{T}} \times K^i \otimes H^3 + H^{3\mathrm{T}} \times K^i \times H^3 \end{aligned} \tag{6.12}$$

Using Eq. (6.11) on the second and third term of Eq. (6.12), we get almost the final result:

$$0 = H^{3\mathrm{T}} \otimes K^i \otimes H^3 - H^{3\mathrm{T}} \times K^i \times H^3 \tag{6.13}$$

Using Eq. (5.4) and Eq. (6.13), we can simplify the 66 component

$$\begin{aligned} H^{3\mathrm{T}} \times K^i \times H^3 \Big|_{66} &= H^3_{66} K^i_{66} H^3_{66} + 2 \underbrace{H^3_{56}}_{=0} K^i_{56} H^3_{66} + \underbrace{H^3_{56}}_{=0} K^i_{55} H^3_{56} \\ &= \left(H^3_{66}\right)^2 K^i_{66} \end{aligned} \tag{6.14}$$

The final results follows:

$$\begin{aligned} H^{3\mathrm{T}} \otimes K^i \otimes H^3 \Big|_{66} &= \left(H^3_{66}\right)^2 K^i_{66} \\ \Longrightarrow \frac{H^{3\mathrm{T}}}{H^3_{66}} \otimes K^i \otimes \frac{H^3}{H^3_{66}} \Big|_{66} &= K^i_{66} \quad \leftarrow \textit{Eq. (6.6) Sands} = \text{Chao} \qquad \text{Q.E.D.} \end{aligned} \tag{6.15}$$