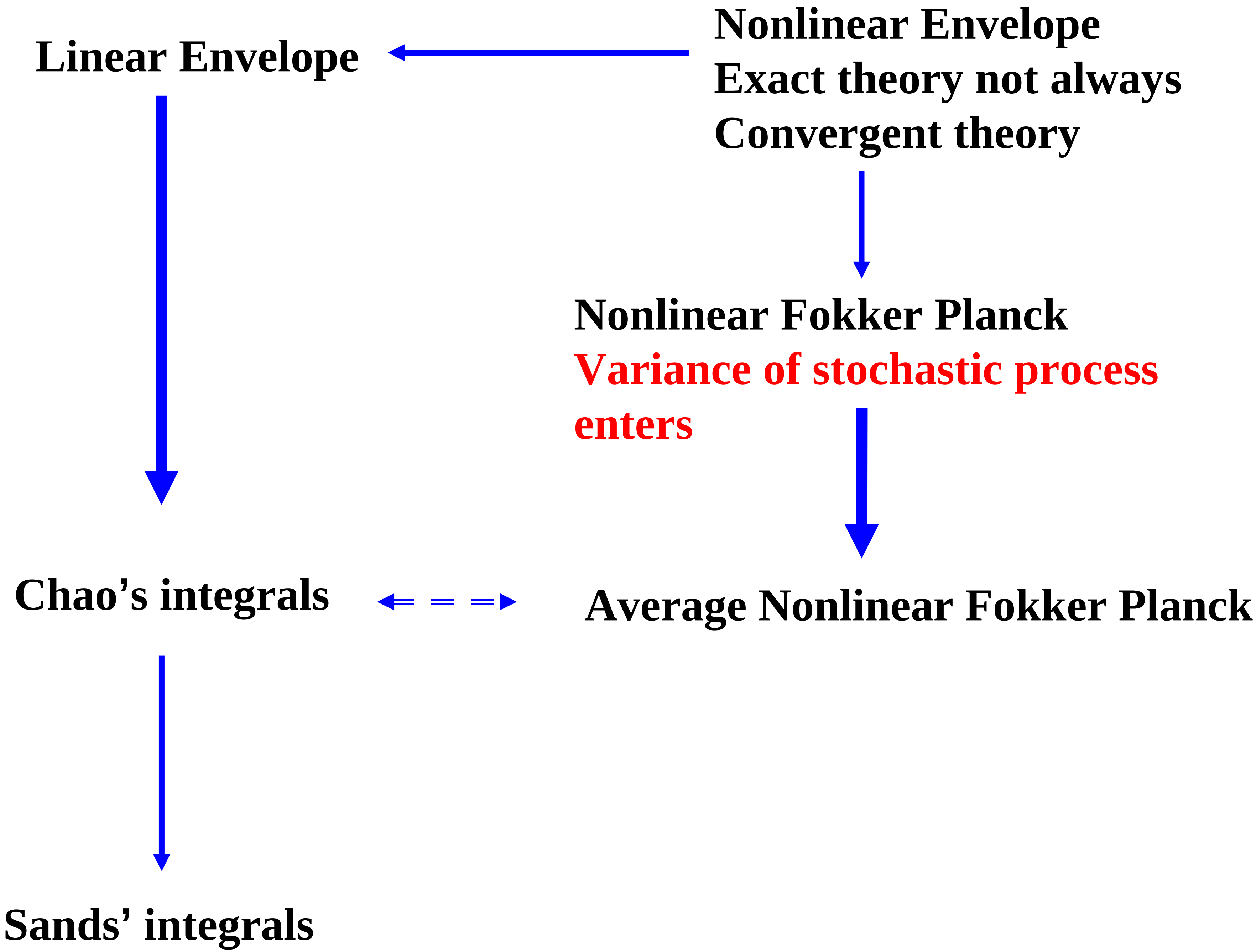
Invariant Dispersive Ripken-like Lattice Functions and from Chao to Sands using these lattice functions
and
Averaging Nonlinear Fokker-Planck

Etienne Forest

KEK

To the memory of Dr. Gerhard Ripken


Linear Envelope
Nonlinear Envelope
Exact theory not always
Convergent theory
Nonlinear Fokker Planck
Variance of stochastic process
enters
Chao's integrals
Average Nonlinear Fokker Planck
Sands' integrals

Topics

- Linear Invariants, linear ergodic relations and Chao/Sands
- How Chao and I got into an argument
- The actual work on averaging Fokker-Planck
- 1 and 2 degrees of freedom theory and examples

Part I
Dispersive Lattice Functions

$$\overline{x}_i = \sum_k A^{-1}_{ik} x_k \tag{83}$$

Now, just for fun, let us suppose that the third plane is frozen. Then the average of Eq. (83), instead of being zero, is given by

$$\langle \overline{\mathbf{x}} \rangle_{1,2} = (0, 0, 0, 0, \overline{x}_5, \overline{x}_6) \tag{84}$$

We then send back this average in the original space using A:

$$\langle x_a \rangle_{1,2} \;=\; \sum_k \left\{ A_{a5} A^{-1}_{5k} + A_{a6} A^{-1}_{6k} \right\} x_k \tag{85}$$

One notices that the lattice functions in Eq. (85) are contractions of A with its inverse. As we said these objects are invariants even in the nonsymplectic case. In the symplectic case, by using the symplectic condition, we can get rid of either A or A^{-1}. Now consider the coefficient of x_5 which we will assume is the energy variable in the longitudinal plane:

$$\begin{aligned} \frac{d\langle x_a \rangle_{1,2}}{dx_5} &= A_{a5} A^{-1}_{55} + A_{a6} A^{-1}_{65} \\ &= A_{a6} A_{66} - A_{a5} A_{56} \end{aligned} \tag{86}$$

From Eq. (86), we see that scalar product invariants between A and its inverse correspond to cross product invariants of A with itself. These cross product invariants do not enter naturally in any perturbative calculation of a system undergoing pseudo-harmonic oscillation. However if one plane is slowly frozen compared to the others, for example the longitudinal plane, these invariant will tend towards dispersive quantities of the constant energy system. For example, the dispersion η is given by the formula

$$\eta_a = A_{a5} A^{-1}_{55} + A_{a6} A^{-1}_{65} \quad \text{for a} = 1, 4 \tag{87}$$

Non-Symplectic Interpretation

$$\vec{z}^{fin} = \vec{z}^{ini} + d\vec{F}$$

$$\text{where } dF_i = \sum_j dF_{ij} z_j .$$

$$\mu_j^{new} = \mu_j + \frac{1}{2} \sum_{a,b} \beta_{ab}^j dF_{ab}$$

$$\alpha_j^{new} = \alpha_j + \frac{1}{2} \sum_{a,b} \eta_{ab}^j dF_{ab} .$$

Pure Mathematical Theory

$$
\begin{aligned}
M &= A \Lambda R A^{-1} \\
&\Downarrow \\
&= A \delta r \Lambda R r^{-1} \delta^{-1} A^{-1}
\end{aligned}
$$

$$
r^{-1}\delta^{-1}A^{-1} = \begin{pmatrix} \delta_1^{-1} r_1^{-1} & 0 & 0 \\ 0 & \delta_2^{-1} r_2^{-1} & 0 \\ 0 & 0 & \delta_3^{-1} r_3^{-1} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A_{11}^{-1} \\ A_{21}^{-1} \end{pmatrix} & \cdots & \begin{pmatrix} A_{16}^{-1} \\ A_{26}^{-1} \end{pmatrix} \\ \vdots & \vdots & \vdots \\ \begin{pmatrix} A_{51}^{-1} \\ A_{61}^{-1} \end{pmatrix} & \cdots & \begin{pmatrix} A_{56}^{-1} \\ A_{66}^{-1} \end{pmatrix} \end{pmatrix}
$$

$$\vec{v}^{i\ j} = \begin{pmatrix} A^{-1}_{2i-1\ j} \\ A^{-1}_{2i\ j} \end{pmatrix},\ i = 1, \cdots, 3$$

$$\delta_i^{-1} r_i^{-1} \vec{v}^{i\ j} = \delta_i^{-1} r_i^{-1} \begin{pmatrix} A^{-1}_{2i-1\ j} \\ A^{-1}_{2i\ j} \end{pmatrix}$$

consider the transpose of the matrix $A\delta r$

which is just $\delta r^{-1} A^t$

$$\delta_i r_i^{-1} \vec{w}^{i\ j} = \delta_i r_i^{-1} \begin{pmatrix} A_{j\ 2i-1} \\ A_{j\ 2i} \end{pmatrix} \quad i = 1, 2, 3.$$

$$\eta^i_{j\ k} = \delta_i^{-1} r_i^{-1} \vec{v}^{i\ j} \cdot \delta_i r_i^{-1} \vec{w}^{i\ k}$$

$$= \vec{v}^{i\ j} \cdot \vec{w}^{i\ k} = A^{-1}_{2i-1\ j} A_{k\ 2i-1} + A^{-1}_{2i\ j} A_{k\ 2i}.$$

$$\beta^i_{j\ k} = \delta_i^{-1} r_i^{-1} \vec{v}^{i\ j} \wedge \delta_i r_i^{-1} \vec{w}^{i\ k}$$

$$= \vec{v}^{i\ j} \wedge \vec{w}^{i\ k} = A^{-1}_{2i-1\ j} A_{k\ 2i} - A^{-1}_{2i\ j} A_{k\ 2i-1}$$

where $\quad (x, y) \wedge (a, b) = x\ b - y\ a.$

Connection to Ripken

In the notation of reference [3], the stroboscopic dispersive averages are all of the form

$$\eta^i_{jk} \;=\; A_{k\ 2i-1} A^{-1}_{2i-1\ j} + A_{k\ 2i} A^{-1}_{2i\ j} \tag{88}$$

and all the standard lattice functions are within a factor of $(-1)^k$ given by

$$\beta^i_{jk} \;=\; \left\{ A^{-1}_{2i-1\ j} A_{k\ 2i} - A^{-1}_{2i\ j} A_{k\ 2i-1} \right\} \tag{89}$$

$$\overline{\beta}^i_{j\overline{k}} \;=\; \left\{ A^{-1}_{2i-1\ j} A^{-1}_{2i-1\ \overline{k}} + A^{-1}_{2i\ j} A^{-1}_{2i\ \overline{k}} \right\} \;=\; (-1)^k \beta^i_{jk} \text{ where } \begin{array}{l} \overline{k} = k+1 \text{ if odd} \\ \overline{k} = k-1 \text{ if even} \end{array}$$

$$I_k \;=\; \frac{1}{2} \sum_{\substack{j,l\,=\,1,6 \\ \varepsilon\,=\,-1,0}} A^{-1}_{2k+\varepsilon\ j} A^{-1}_{2k+\varepsilon\ l}\ x_j x_l = \frac{1}{2} \sum_{j,l=1,6} \overline{\beta}^k_{jl}\ x_j x_l$$

Ripken Functions

Beam Envelope

$$\Sigma_{kl} \; = \; \langle x_k x_l \rangle$$

$$\Sigma^f \; = \; M \Sigma^i \widetilde{M} \; + \; \Delta\Sigma$$

$$\overline{\Sigma}^f \; = \; \Lambda \overline{\Sigma}^i \Lambda + \overline{\Delta\Sigma}$$

$$M = C \Lambda C^{-1} \text{ and } \overline{\Sigma} = C \Sigma \widetilde{C}$$

$$\Lambda = diag\left(\exp\left(-\alpha_1 + i\mu_1\right), \exp\left(-\alpha_1 - i\mu_1\right), \cdots, \exp\left(-\alpha_3 + i\mu_3\right), \exp\left(-\alpha_3 - i\mu_3\right)\right)$$

$$\langle \overline{x}_{2k-1} \overline{x}_{2k} \rangle \; = \; \overline{q}_k^2 + \overline{p}_k^2$$

From Envelope to Chao to Sans

Thus we can immediately solve for the equilibrium phasors. Consider first the terms $\langle \overline{x}_{2k-1}\overline{x}_{2k}\rangle = \overline{q}_k^2 + \overline{p}_k^2$ where $k = 1, 2, 3$:

$$\langle \overline{x}_{2k-1}\overline{x}_{2k}\rangle^{\infty} = \frac{\overline{\Delta\Sigma}_{2k-1\ k}}{1 - \exp\left(-2\alpha_k\right)} \qquad (105)$$

There are three terms of the type of Eq. (105). They are the so-called equilibrium emittances. If we look at a different generic term

$$\langle \overline{x}_1\overline{x}_3\rangle^{\infty} = \frac{\overline{\Delta\Sigma}_{13}}{1 - \exp\left[-\alpha_1 - \alpha_2 + i\left(\mu_1 + \mu_2\right)\right]} \qquad (106)$$

This is the equilibrium phasor corresponding to the $\nu_x + \nu_y$ resonance. It is clear that if the damping decrements are small compared to the distance to a linear resonance, i.e.,

$$|\alpha_i \pm \alpha_j| \ll |1 - \exp\left[i\left(\mu_i \pm \mu_j\right)\right]| \qquad (107)$$

$$\Delta\left\{q_k^2+p_k^2\right\}_{total} = \oint_0^C \left\{ {A_{2k-1\ 5}^{-1}}^2 + {A_{2k\ 5}^{-1}}^2 \right\} \frac{d\left\langle \Delta_5^2 \right\rangle_{radiation}}{ds} ds$$

$$= \oint_0^C \overline{\beta}_{k5}^3 \frac{d\left\langle \Delta_5^2 \right\rangle_{radiation}}{ds} ds$$

Chao

???? if k=1

Sands

$$\varepsilon_1 = \left(\sum_{j=1,4} A_{1j}^{-1} \eta_{j5}^3 \right)^2 + \left(\sum_{j=1,4} A_{2j}^{-1} \eta_{j5}^3 \right)^2$$

$$= \sum_{k,j=1,4} \overline{\beta}_{kj}^1 \eta_{k5}^3 \eta_{j5}^3$$

$$\sum_{j=1,4} A^{-1}_{1j}\eta^6_{j5} = \sum_{j=1,4} A^{-1}_{1j}A_{j5}A^{-1}_{55} + A^{-1}_{1j}A_{j6}A^{-1}_{65}$$
$$= -\left\{A^{-1}_{15}A_{55} + A^{-1}_{16}A_{65}\right\}A^{-1}_{55} - \left\{A^{-1}_{15}A_{56} + A^{-1}_{16}A_{66}\right\}A^{-1}_{65}$$
$$= -A^{-1}_{15}\left\{A_{55}A^{-1}_{55} + A_{56}A^{-1}_{65}\right\} - A^{-1}_{16}\left\{A_{65}A^{-1}_{55} + A_{66}A^{-1}_{65}\right\}$$
$$= -A^{-1}_{15}\eta^3_{55} - A^{-1}_{16}\eta^3_{65}$$

$$\sum_{j=1,4} A^{-1}_{2j}\eta^6_{j5} = -A^{-1}_{25}\eta^3_{55} - A^{-1}_{26}\eta^3_{65}$$

Final Result entirely in terms

of Standard and Dispersive Invariant Ripken lattice functions

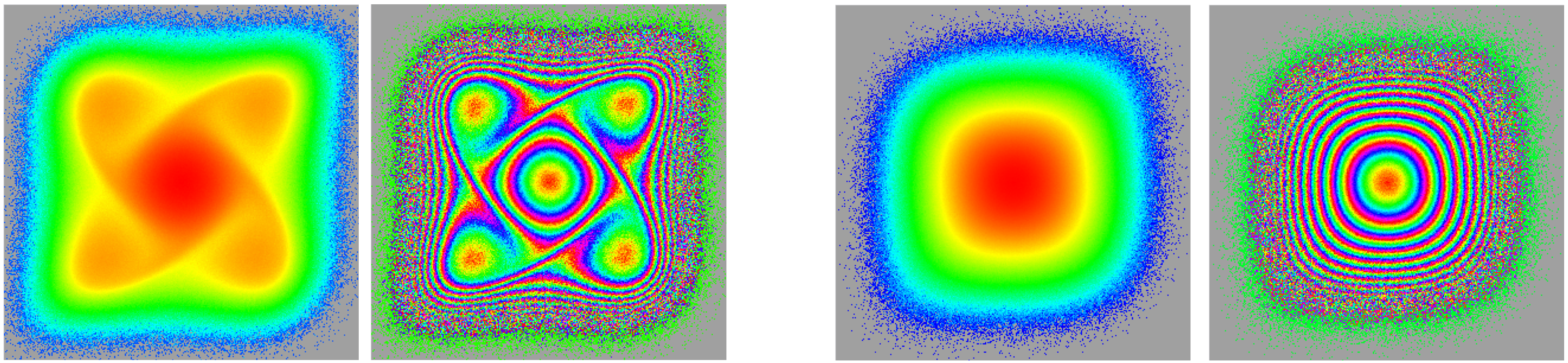
$$\varepsilon_1 = \left\{{A^{-1}_{15}}^2 + {A^{-1}_{25}}^2\right\}{\eta^3_{55}}^2 + \left\{{A^{-1}_{16}}^2 + {A^{-1}_{26}}^2\right\}\eta^3_{65} + 2\left\{A^{-1}_{15}A^{-1}_{16} + A^{-1}_{26}A^{-1}_{26}\right\}\eta^3_{55}\eta^3_{65}$$
$$\Downarrow$$
$$\left\{{A^{-1}_{15}}^2 + {A^{-1}_{25}}^2\right\} = \frac{1}{{\eta^3_{55}}^2}\left[\varepsilon_1 - \left\{{A^{-1}_{16}}^2 + {A^{-1}_{26}}^2\right\}\eta^3_{65} - \left\{A^{-1}_{15}A^{-1}_{16} + A^{-1}_{26}A^{-1}_{26}\right\}\eta^3_{55}\eta^3_{65}\right]$$
$$= \frac{1}{{\eta^3_{55}}^2}\left[\varepsilon_1 - \overline{\beta}^1_{66}\eta^3_{65} - \overline{\beta}^1_{56}\eta^3_{55}\eta^3_{65}\right]$$

Part II
Non-Linear Averaged Fokker-Planck

Chao's One-Resonance K

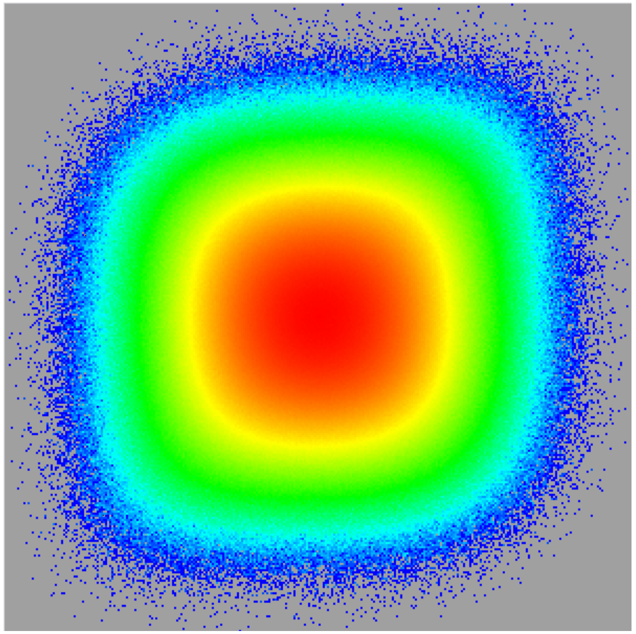
$$K = \frac{1}{R}\left[\left(\nu - \frac{m}{n}\right) J + D_\nu (J) + f_1 (\phi, J)\right]$$

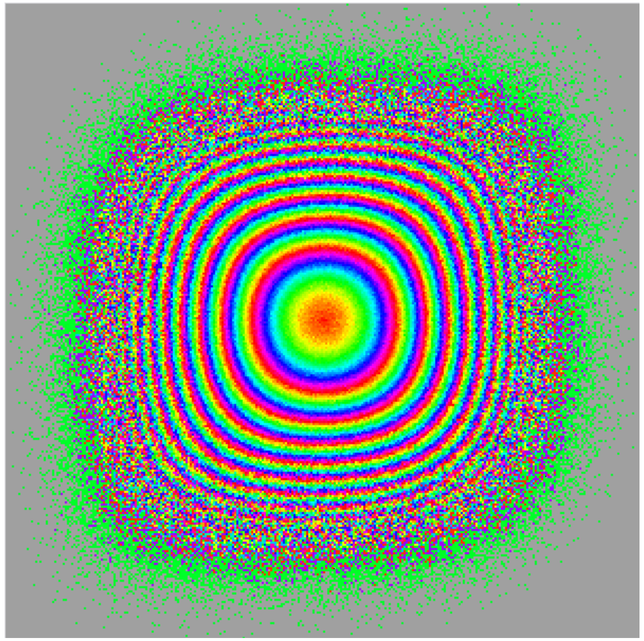
where $$f_1\left(\phi + \frac{2\pi}{n}, J\right) = f_1 (\phi, J)$$



$$K = \frac{1}{R}\left[\left(\nu - \frac{m}{n}\right)J + D_\nu(J) + f_1(\phi, J)\right]$$

$$\overline{K}(I) = \frac{1}{R}\left[\left(\nu - \frac{m}{n}\right)I + \overline{D}_\nu(I)\right]$$





Chao's Falsely Resonant Result

$$\psi(\phi, J) \;\propto\; \exp\left[-\frac{1}{J_0}\left(J + \frac{f_1(\phi, J)}{\nu - \frac{m}{n} + D'_\nu(J)}\right)\right]$$

But Etienne says that this is a trivial non-resonant result

$$I \;=\; J + \frac{f_1(\phi, J)}{\nu - \frac{m}{n} + D'_\nu(J)} \;+\; O\left(||f_1||^2\right)$$

$$\psi(\phi, J) \;\propto\; \exp\left(-\frac{I(\phi, J)}{J_0}\right)$$

Hypotheses

$$m_s = a_s \circ r \circ a_s^{-1} \quad K = K(J;s) \text{ where } J_i = \frac{q_i^2 + p_i^2}{2}$$

$$m_s^r = a_s^r \circ \rho \circ a_s^{r\ -1}$$

$$\rho \quad \Rightarrow \quad \exp\left(\mu \cdot \partial_\psi\right)\ \exp\left(-a \cdot \partial_I\right)$$

$$\exp\left(\mu \cdot \partial_\psi\right)\ \exp\left(-a \cdot \partial_I\right) I \ = \ \exp(-2\alpha(I))I$$

$$I_i = J_i \circ a^{-1}$$

Deriving Averaged Fokker-Planck

$$\int f(I)\ dI\ d\psi \ = \ \int f(I(\overline{I}))\ \left|\frac{dI\ d\psi}{d\overline{I}\ d\overline{\psi}}\right|\ d\overline{I} d\overline{\psi}$$

$$\begin{aligned}\psi &= \exp\left(-\mu\cdot\partial_{\overline{\psi}}\right)\ \exp\left(a\cdot\partial_{\overline{I}}\right)\overline{\psi} \\ &= \overline{\psi}-\mu\cdot\overline{I}\end{aligned}$$

$$I_k \ = \ e^{2\alpha_k(I)}\overline{I}_k \ = \ \overline{I}_k + 2\alpha_k(\overline{I})\overline{I}_k \ + \ |\alpha|^2\cdots$$

$$\int f(I)\ dI\ d\psi \quad = \quad \int f(I(\bar{I}))\ \left|\frac{dI\ d\psi}{d\bar{I}\ d\bar{\psi}}\right|\ d\bar{I}d\bar{\psi}$$

$$\left|\frac{dI\ d\psi}{d\bar{I}\ d\bar{\psi}}\right| \quad = \ 1 + 2\sum_k \frac{\partial\alpha_k(\bar{I})\bar{I}_k}{\partial\bar{I}_k}\ +\ |\alpha|^2\cdots$$

$$\Delta f \quad = \ 2\sum_k \{\alpha_k I_k \partial_k f\ + f\ \partial_k \alpha_k I_k\}\cdots$$

Fluctuation Effects

$$S(x_k) = x_k + \Delta_k \ , \ \frac{\partial \Delta_k}{\partial x_a} = 0 \quad \text{and} \quad \langle \Delta_k \rangle = 0$$

$$f\left(I(\overline{I})\right) \quad = \quad f\left(I - \sum_a \frac{\partial}{\partial x_a} I \Delta_a + \frac{1}{2} \sum_{a,b} \frac{\partial^2}{\partial x_a \partial x_b} I \Delta_a \Delta_b \right)$$

$$= \ f + \sum_{k,a} \frac{\partial f}{\partial I_k} \left\{ - \ \frac{\partial I_k}{\partial x_a} \Delta_a \right\}$$

$$+ \sum_{k,a,b} \frac{\partial f}{\partial I_k} \left\{ \frac{1}{2} \frac{\partial^2 I_k}{\partial x_a \partial x_b} \Delta_a \Delta_b \right\} + \sum_{k,m,a,b} \left\{ \frac{1}{2} \frac{\partial^2 f}{\partial I_k \partial I_m} \ \frac{\partial I_k}{\partial x_a} \frac{\partial I_m}{\partial x_b} \Delta_a \Delta_b \right\}$$

Averaging

$$\Delta f \;=\; \sum_{k,a,b} \frac{\partial f}{\partial I_k} \left\{ \frac{1}{2} \frac{\partial^2 I_k}{\partial x_a \partial x_b} \sigma_{ab} \right\} + \sum_{k,m,a,b} \frac{1}{2} \frac{\partial^2 f}{\partial I_k \partial I_m} \frac{\partial I_k}{\partial x_a} \frac{\partial I_m}{\partial x_b} \sigma_{ab}$$

where $\sigma_{ab} \;=\; \langle \Delta_a \Delta_b \rangle_{\text{stochastic process}}$

$$\Delta f \;=\; \sum_{k,a,b} \frac{\partial f}{\partial I_k} \left\{ \frac{1}{2} \left\langle \frac{\partial^2 I_k}{\partial x_a \partial x_b} \sigma_{ab} \right\rangle \right\} + \sum_{k,m,a,b} \frac{1}{2} \frac{\partial^2 f}{\partial I_k \partial I_m} \left\langle \frac{\partial I_k}{\partial x_a} \frac{\partial I_m}{\partial x_b} \sigma_{ab} \right\rangle$$

$$\frac{\Delta f}{\Delta n} \;=\; 2 \sum_{k} \frac{\partial}{\partial I_k} \{\alpha_k I_k f\} + \sum_{k,a,b} \frac{\partial f}{\partial I_k} \left\{ \frac{1}{2} \left\langle \frac{\partial^2 I_k}{\partial x_a \partial x_b} \sigma_{ab} \right\rangle \right\} + \sum_{k,m,a,b} \frac{1}{2} \frac{\partial^2 f}{\partial I_k \partial I_m} \left\langle \frac{\partial I_k}{\partial x_a} \frac{\partial I_m}{\partial x_b} \sigma_{ab} \right\rangle$$

Deterministic

Stochastic

Theorem (Conjecture)

$$\left\langle \frac{\partial^2 I_k}{\partial x_a \partial x_b} \right\rangle = \sum_m \frac{\partial}{\partial I_m} \underbrace{\frac{1}{2} \left\langle \left\{ \frac{\partial I_k}{\partial x_a} \frac{\partial I_m}{\partial x_b} + \frac{\partial I_m}{\partial x_a} \frac{\partial I_k}{\partial x_b} \right\} \right\rangle}_{\Gamma_{kmab}} \quad (23)$$

This theorem appears to be true for all possible maps "a", including non-symplectic maps, such that

$$I_i = J_i \circ a^{-1}$$

Consequence

$$
\begin{aligned}
\frac{\Delta f}{\Delta n} &= \sum_k \frac{\partial}{\partial I_k}\{2\alpha_k I_k f\} + \frac{1}{2}\sum_{k,m,a,b}\left\{\frac{\partial^2 f}{\partial I_k \partial I_m}\Gamma_{kmab} + \frac{\partial f}{\partial I_k}\;\frac{\partial}{\partial I_m}\Gamma_{kmab}\right\} \\
&= \sum_k \frac{\partial}{\partial I_k}\left\{2\alpha_k I_k f \;+\; \frac{1}{2}\sum_{m,a,b}\frac{\partial f}{\partial I_m}\Gamma_{kmab}\right\} \qquad (24)
\end{aligned}
$$

Lemma

$$\Gamma_{kk\,ab} = I_k\ D_{kk\,ab} \tag{25}$$

$$\Gamma_{km\,ab} = I_k I_m\ D_{km\,ab} \text{if } k \neq m \tag{26}$$

$$D_{km\,ab} \text{ is a polynomial in the I's}$$

The properties are simple consequences of the averaging process for an arbitrary canonical transformation a in the nonlinear case. As for the linear part, the off diagonal terms of Γ actually vanish. This is actually consistent with the Chao[1] theory of synchrotron integrals.
The diffusion terms D will enter in the general solution more naturally than Γ.

Theorem : equilibrium is a quadrature

Using Eq. (24), we get an equation for the equilibrium distribution by setting $\frac{\Delta f}{\Delta n} = 0$:

$$2\alpha f + \frac{1}{2} D \frac{\partial f}{\partial I} = \text{constant} = 0 \tag{27}$$

We select the constant in Eq. (27) to be zero simply because the average of I is not defined otherwise. The diffusion D is given by:

$$D = D_{11\,11}\sigma_{11} + D_{11\,22}\sigma_{22} + 2D_{11\,12}\sigma_{12} \tag{28}$$

Thus we get for the equilibrium distribution:

$$\begin{aligned} f(I) &= \lambda^{-1} \exp\left(- \int_0^I \frac{4\alpha}{D} dI \right) \\ \lambda &= \int_0^\infty \exp\left(- \int_0^I \frac{4\alpha}{D} dI \right) dI \end{aligned} \tag{29}$$

Here we will look at a simple example which is completly solvable. We invite the reader to do the simulation. Consider the following map for a:

$$\begin{aligned} a_1^{-1} &= q \\ a_2^{-1} &= p - c\, q^2 \end{aligned} \tag{31}$$

For the rotation we choose a simple linear rotation in phase space:

$$\begin{aligned} r_1 &= \cos\mu\, q &+& \sin\mu\, p \\ r_2 &= -\sin\mu\, q &+& \cos\mu\, p \end{aligned} \tag{32}$$

So the total Hamiltonian map is just given by $m = a \circ r \circ a^{-1}$. We follow this map by a "radiative" stochastic map of the form:

$$\begin{aligned} s &= D \circ \Lambda \\ \Lambda(q,p) &= (\lambda q, \lambda p) \\ D(x,p) &= (x + \sigma_{11}\xi_1, p + \sigma_{22}\xi_2) \end{aligned} \tag{33}$$

The variables ξ_1 and ξ_2 are uncorrelated stochastic random variables of average zero and variance one. The total map is thus:

$$m = D \circ \Lambda \circ a \circ r \circ a^{-1} \quad (34)$$

The invariant can be easily computed using a^{-1}:

$$I(x,p) = \frac{x^2 + (p - cx)^2}{2} \quad (35)$$

When can now evaluate the average fluctuation of I over the random processes r_1 and r_2:

$$\langle I(q,p)\rangle_{process} \;=\; I(q,p) + \frac{\sigma_{11}^2 + \sigma_{22}^2}{2} - cp\sigma_{11}^2 + 3c^2\sigma_{11}^2 q^2 \tag{36}$$

We average over the nonlinear canonical phase, this is done by substituting a and doing an average in the nonlinear Floquet variables:

$$\begin{aligned}\Big\langle \langle I(q,p)\rangle_{process} \circ a \Big\rangle_{phase} \;&=\; J + \frac{\sigma_{11}^2 + \sigma_{22}^2}{2} - c\underbrace{\big\langle (p + cq^2)\big\rangle_{phase}}_{J}\sigma_{11}^2 + 3c^2\sigma_{11}^2\underbrace{\big\langle q^2\big\rangle_{phase}}_{J} \\ &=\; J + \frac{\sigma_{11}^2 + \sigma_{22}^2}{2} + 2c^2\sigma_{11}^2 J \end{aligned}\tag{37}$$

Of course, we bring this back in the original (q,p) using a^{-1}:

$$\langle \Delta I\rangle \;=\; \left(\Big\langle \langle I(q,p)\rangle_{process} \circ a \Big\rangle_{phase} - J\right) \circ a^{-1} \;=\; \frac{\sigma_{11}^2 + \sigma_{22}^2}{2} + 2c^2\sigma_{11}^2 I \tag{38}$$

Using Eq. (23), i.e.,

$$2\langle \Delta I \rangle = \frac{\partial}{\partial I} I D \tag{39}$$

we get for D:

$$D = \underbrace{\sigma_{11}^2 + \sigma_{22}^2}_{D_0} + \underbrace{2c^2\sigma_{11}^2}_{D_1} I \tag{40}$$

We now solve Eq. (29),

$$f(I) = \left\{ \frac{4\alpha}{D_0} - \frac{D_1}{D_0} \right\} \left\{ 1 + \frac{D_1}{D_0} I \right\}^{-4\alpha/D_1} \tag{41}$$

The population from $I = 0$ to I can be computed:

$$P(I) = \int_0^I f(I) dI = 1 - \left\{ 1 + \frac{D_1}{D_0} I \right\}^{1-4\alpha/D_1} \tag{42}$$

The average value of I is given by:

$$\langle I \rangle = \frac{D_0}{D_1 \left\{ \frac{4\alpha}{D_1} - 2 \right\}} \tag{43}$$

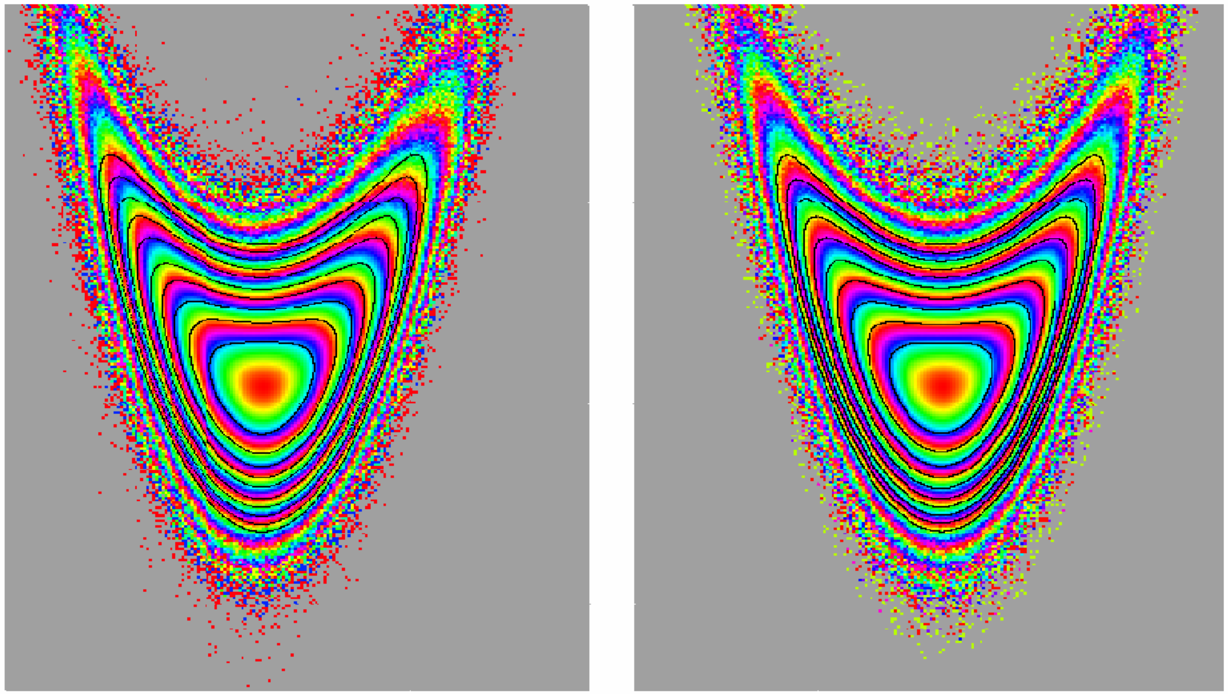
$$c = 0.2$$

$$\nu = \frac{\mu}{2\pi} = 0.2442$$

$$\lambda = 0.99 \text{ or } 0.999$$

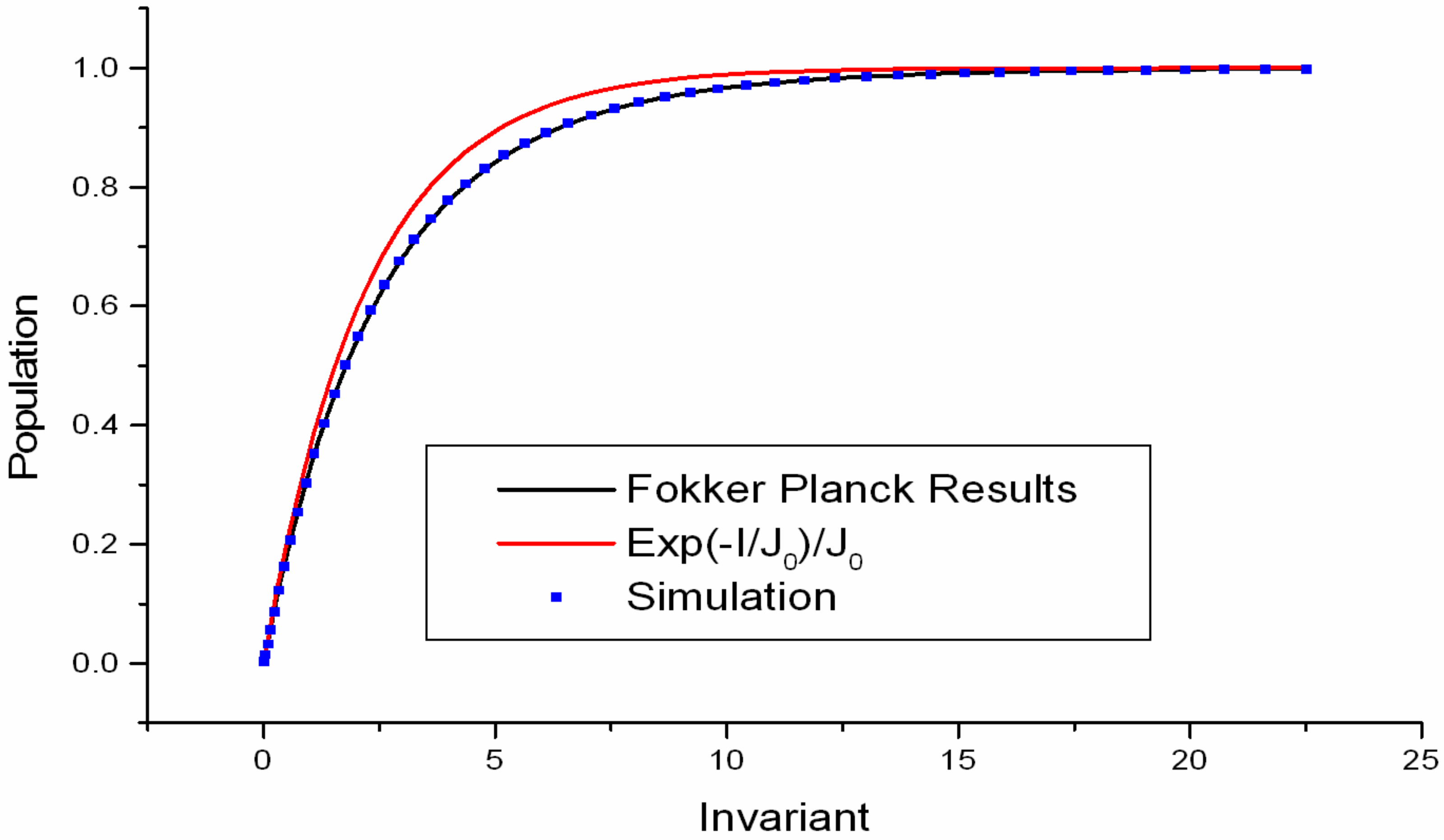
$$J_0 = 2.25 \;\Rightarrow\; \sigma_{11} = \sigma_{22} = \sqrt{(1-\lambda^2)\, J_0}$$

$$\alpha = \frac{1}{2}\left(1-\lambda^2\right)$$



Distribution on a Logarithmic Rainbow Scale: $\lambda = 0.99$ and $\lambda = 0.999$

Results of 1-d-of-f



Plot of the Integrated Population

2 and 3 degrees of freedom

Things look more complex in several degrees of freedom. We will again assume that the distribution is an exponential and see what mathematical tricks we need to pull to get a friendly quadrature. We first notice that the turn derivative df/dn is of the form

$$\frac{\Delta f}{\Delta n} = \nabla \cdot v = 0 \tag{52}$$

where the vector v is just:

$$v_k = 2\alpha_k I_k f + \frac{1}{2} \sum_{m,a,b} \frac{\partial f}{\partial I_m} \Gamma_{kmab} \tag{53}$$

Curl of a function

Let us assume, incorrectly as it will turn out, that $v = 0$ for the equilibrium distribution. We write $f = e^{\theta}$ and get for θ:

$$2\alpha_k I_k \; + \frac{1}{2}\sum_{m,a,b} \frac{\partial\theta}{\partial I_m}\Gamma_{kmab} \; = \; 0 \qquad (54)$$

We can use properties (25) and (26) to replace Γ by D. Then we can invert D to solve for the one-form $d\theta$ as a function of D^{-1} and the damping decrements. All of this will make sense if and only if the one-form $d\theta$ is indeed the derivative of a function. Now the gods are abandoning us after all. **It is not the case in general!** In fact we have

$$\exists\Phi(I) \;\; \rightarrow \;\; 2\alpha_k I_k \; + \frac{1}{2}\sum_{m,a,b} \frac{\partial\theta}{\partial I_m}\Gamma_{kmab} \;\; = \;\; -\nabla \wedge \Phi \quad (55)$$

We can start with the case $N = 2$. Let us introduce a gauge function $\Phi = \frac{1}{2} I_1 I_2 f \Psi$. The equilibrium distribution is then given by the equations:

$$
\begin{aligned}
0 &= 2\alpha_1 I_1 f \; + \frac{1}{2}\left[I_1 D_{11} \frac{\partial f}{\partial I_1} + \; I_1 I_2 D_{12} \frac{\partial f}{\partial I_2} + \frac{\partial}{\partial I_2}\left\{ I_1 I_2 f \Psi \right\} \right] \\
0 &= 2\alpha_2 I_2 f \; + \frac{1}{2}\left[I_2 D_{22} \frac{\partial f}{\partial I_2} + \; I_1 I_2 D_{12} \frac{\partial f}{\partial I_1} - \frac{\partial}{\partial I_1}\left\{ I_1 I_2 f \Psi \right\} \right]
\end{aligned} \tag{56}
$$

Expanding all of this and expressing Eq. (56) in terms of θ, we get:

$$
\begin{aligned}
0 &= \left\{ 4\alpha_1 + \frac{\partial}{\partial I_2}\left\{ I_2 \Psi \right\} \right\} + \underbrace{D_{11}}_{D'_{11}} \frac{\partial \theta}{\partial I_1} + \underbrace{\left\{ \, I_2 D_{12} + I_2 \Psi \right\}}_{D'_{12}} \frac{\partial \theta}{\partial I_2} \\
0 &= \left\{ 4\alpha_2 - \frac{\partial}{\partial I_1}\left\{ I_1 \Psi \right\} \right\} + \underbrace{D_{22}}_{D'_{22}} \frac{\partial \theta}{\partial I_2} + \underbrace{\left\{ \, I_1 D_{12} - I_1 \Psi \right\}}_{D'_{21}} \frac{\partial \theta}{\partial I_1}
\end{aligned} \tag{57}
$$

$$\begin{pmatrix} \frac{\partial\theta}{\partial I_1} \\ \frac{\partial\theta}{\partial I_2} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \equiv \begin{pmatrix} {D'_{21}}^{-1}\left\{4\alpha_1 + \frac{\partial}{\partial I_2}\{I_2\Psi\}\right\} + {D'_{22}}^{-1}\left\{4\alpha_2 - \frac{\partial}{\partial I_1}\{I_1\Psi\}\right\} \\ {D'_{21}}^{-1}\left\{4\alpha_1 + \frac{\partial}{\partial I_2}\{I_2\Psi\}\right\} + {D'_{22}}^{-1}\left\{4\alpha_2 - \frac{\partial}{\partial I_1}\{I_1\Psi\}\right\} \end{pmatrix} \tag{58}$$

The equation for Ψ is gotten by insuring that the RHS of Eq. (58) is the derivative of a function:

$$0 = \frac{\partial F_1}{\partial I_2} - \frac{\partial F_2}{\partial I_1} \tag{59}$$

Eq. (59) will produce, for a generic coupled system, an equation which requires a function Ψ which is of the form $\Psi = \Psi_0 + \Psi_{1;0} I_1 + \Psi_{0;1} I_2 + \cdots$, that is to say, a polynomial in the invariants. The Taylor coefficient of Ψ, like the moments of a distribution, are affected by the higher coefficients. This means that we must solve for these all at once using a Newton search. It should be added that there is a unique Ψ for mildly coupled systems near the linear/decoupled solution $\Psi = 0$. Once the function Ψ is obtained, then the equilibrium distribution f is a quadrature:

$$\theta = \underbrace{\iint_0^I F_1 dI_1 + F_2 dI_2}_{\text{Arbitrary Path from 0 to } I} \tag{60}$$

IV. CONCLUSION AND FUTURE WORK

We have seen that

- if the fluctuations of the Cartesian variables can be lumped at one place in the ring,
- if only the constant part of these fluctuations are kept,
- if damping decrements are small compared to the linear resonances,
- if the map is fully normalizable,

then it is possible to solve for the equilibrium distribution by quadrature.
In the future, this work will be complete if the following related problems can be solved.

1. Obtain an averaged Fokker-Planck equation in the presence of position dependent fluctuations. This will have to include the necessary ergodic conjectures.

2. Study the commutation rules between deterministic operators and diffusion operators so that the s-dependent effects can be correctly lumped at one place. This will produce position dependent diffusion operators even if the individual fluctuations at all positions s are linear, requiring the results of item 1.