WHAT'S SO GREAT ABOUT KAM THEORY?

H. S. Dumas
U. Cincinnati

Hamiltonian Systems
("apotheosis of mathematical models of classical mechanics")

A Hamiltonian system (H, M) of n degrees of freedom consists of a $2n$-dimensional phase space M with conjugate canonical coordinates $(p, q) = (p_1, \dots, p_n; q_1, \dots, q_n)$ and a smooth Hamiltonian function $H : M \to \mathbb{R}$ which generates $2n$ first-order differential equations (the canonical eqns. of motion):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, \dots, n)$$

The set of all solns. of these eqns. is called the (phase) flow of H on M.

Given two smooth functions $F : M \to \mathbb{R}$, $G : M \to \mathbb{R}$, their Poisson bracket is

$$\{F, G\} = \frac{\partial F}{\partial p} \cdot \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \cdot \frac{\partial G}{\partial p} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right)$$

Completely Integrable Hamiltonian Systems
(Classically solvable systems)

The n degree of freedom Ham. system (H, M) is **completely integrable** if it has n independent **first integrals** (or "constants of motion") $F_1, \dots, F_n$ $(F_i : M \to \mathbb{R})$ **in involution** $(\{F_i, F_j\} = 0)$.

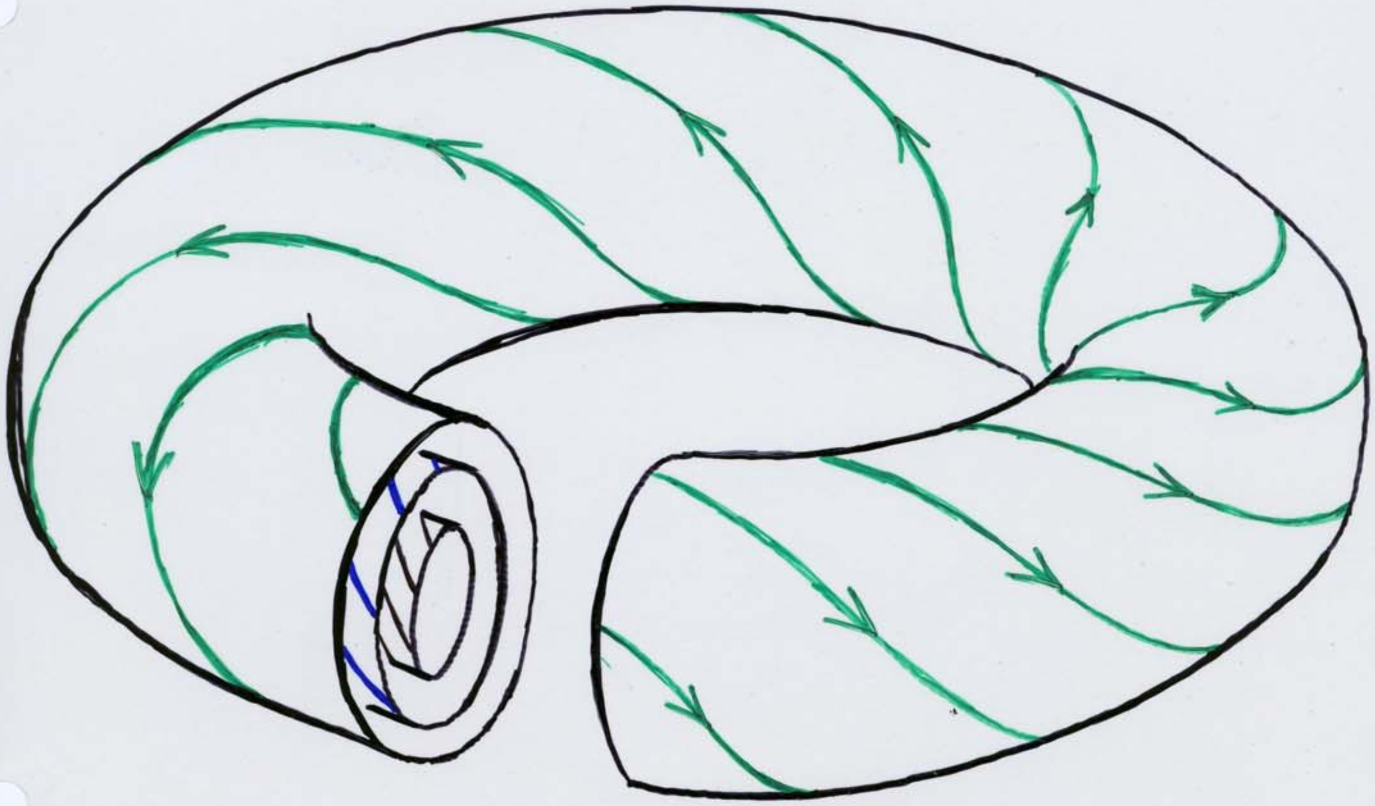
Level sets S obtained by setting the F_i = const. are smooth invariant n-dimensional hypersurfaces in M; in fact whenever the S are bounded, they are invariant **n-tori** $(S = \mathbb{T}^n)$ and there are special canonical coordinates $(I, \theta) = (I_1, \dots, I_n; \theta_1, \dots, \theta_n)$ on M (called **action-angle variables**) in which H is a function of I only, so the eqns. of motion

$$\frac{d\theta_i}{dt} = \omega_i(I), \quad \frac{dI_i}{dt} = 0 \qquad \left(\text{frequency } \omega_i(I) = \frac{\partial H}{\partial I_i}\right)$$

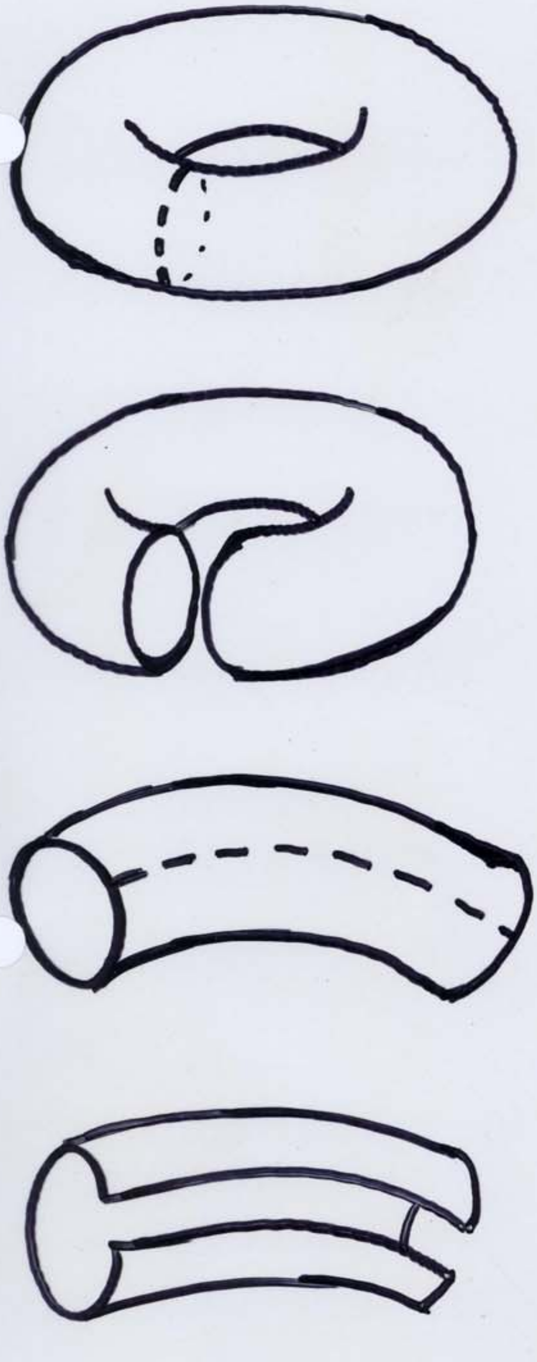
have solns:

$$\theta_i(t) = \omega_i(I^0)\, t + \theta_i^0, \quad I_i(t) = I_i^0$$

(i.e. **linear flow** $\theta(t) = \omega t + \theta^0$ with fixed frequency $\omega = \omega(I^0)$ on the invariant n-torus $I = I^0$)



Invariant tori of completely integrable Hamiltonian system (2 degrees of freedom)

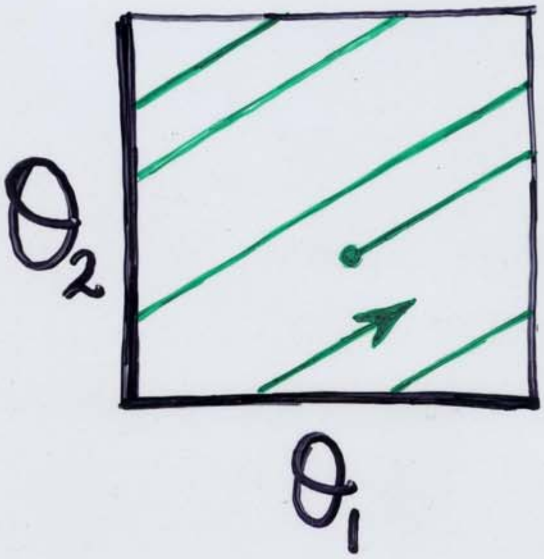


Linear flow

$$\theta(t) = \omega t + \theta^0$$

on $\mathbb{T}^2$

$$\text{Slope} = \frac{\omega_2}{\omega_1}$$

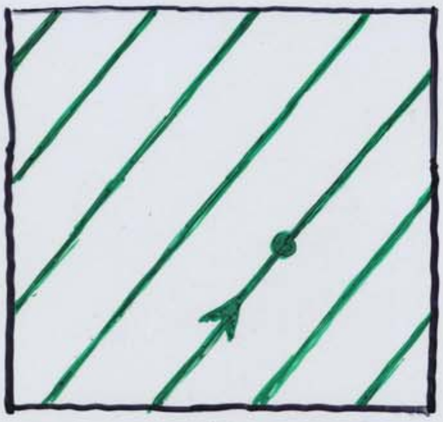


Resonant versus Nonresonant Tori

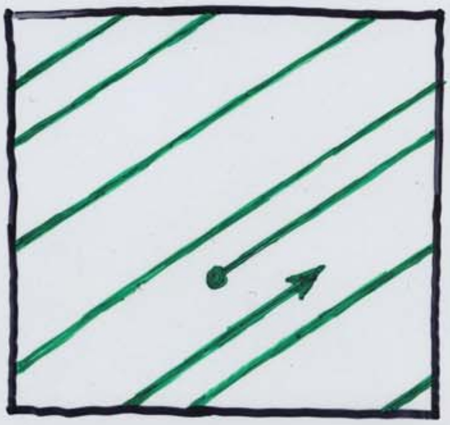
For fixed frequency $\omega = (\omega_1, \dots, \omega_n)$ there are two possible cases:

RESONANT CASE	**NONRESONANT CASE**
$0 = k \cdot \omega = k_1\omega_1 + \dots + k_n\omega_n$ for some nonzero integer vector $k = (k_1, \dots, k_n)$	There is NO nonzero integer vector $k = (k_1, \dots, k_n)$ such that $0 = k \cdot \omega = k_1\omega_1 + \dots + k_n\omega_n$

Pictures for $n = 2$ (2-dimensional tori):



slope $\frac{\omega_2}{\omega_1}$ rational
Closed orbits



slope $\frac{\omega_2}{\omega_1}$ irrational
Orbits do not close, but fill torus densely

For "nondegerate" systems, resonant & nonresonant tori are coencentrically interlaced in phase space in the same way that rational & irrational numbers are interlaced in the real numbers.

Kolmogorov (1954)
Arnold (1963)
Moser (1962)

Prototype KAM theorem

Let $(I,\theta) = (I_1,\dots,I_n;\theta_1,\dots,\theta_n)$ be action-angle variables for the smooth completely integrable Hamiltonian (h, M) [with $n \geq 2$ degs. of fr.]. Assume h is nondegenerate.

Then there exists (a small) $\varepsilon > 0$ such that whenever $P = P(I,\theta)$ is a smooth perturbation of size $|P|_M < \varepsilon$, the perturbed Hamiltonian

$$H(I,\theta) = h(I) + P(I,\theta)$$

has a nonempty set $\mathcal{T}$ of n-dimensional invariant tori in its phase space. On each invariant torus of $\mathcal{T}$, the flow of H is quasiperiodic (i.e., linear with [highly] nonresonant frequency).

SYNTHESIS !

Ramifications & Consequences of KAM

- The half-century-old paradox/crisis of perturbation theory in celestial mechanics is resolved: both Weierstraß & Poincaré are right (but Weierstraß moreso — the Lindstedt series <u>are</u> convergent).

- New qualitative, geometric methods supersede quantitative approaches based on classical integrability (classical integrability is immediately destroyed by perturbation; but a more general geometric kind of integrable behavior persists).

- The solar system is stable! (well, at least certain Hamiltonian models of it...).

- In classical statistical mechanics & kinetic theory, (the descendant of) Boltzmann's ergodic hypothesis ["Ergodensatz"] is dealt a harsh blow ("Generic Hamiltonian systems are neither integrable nor ergodic").

PRO

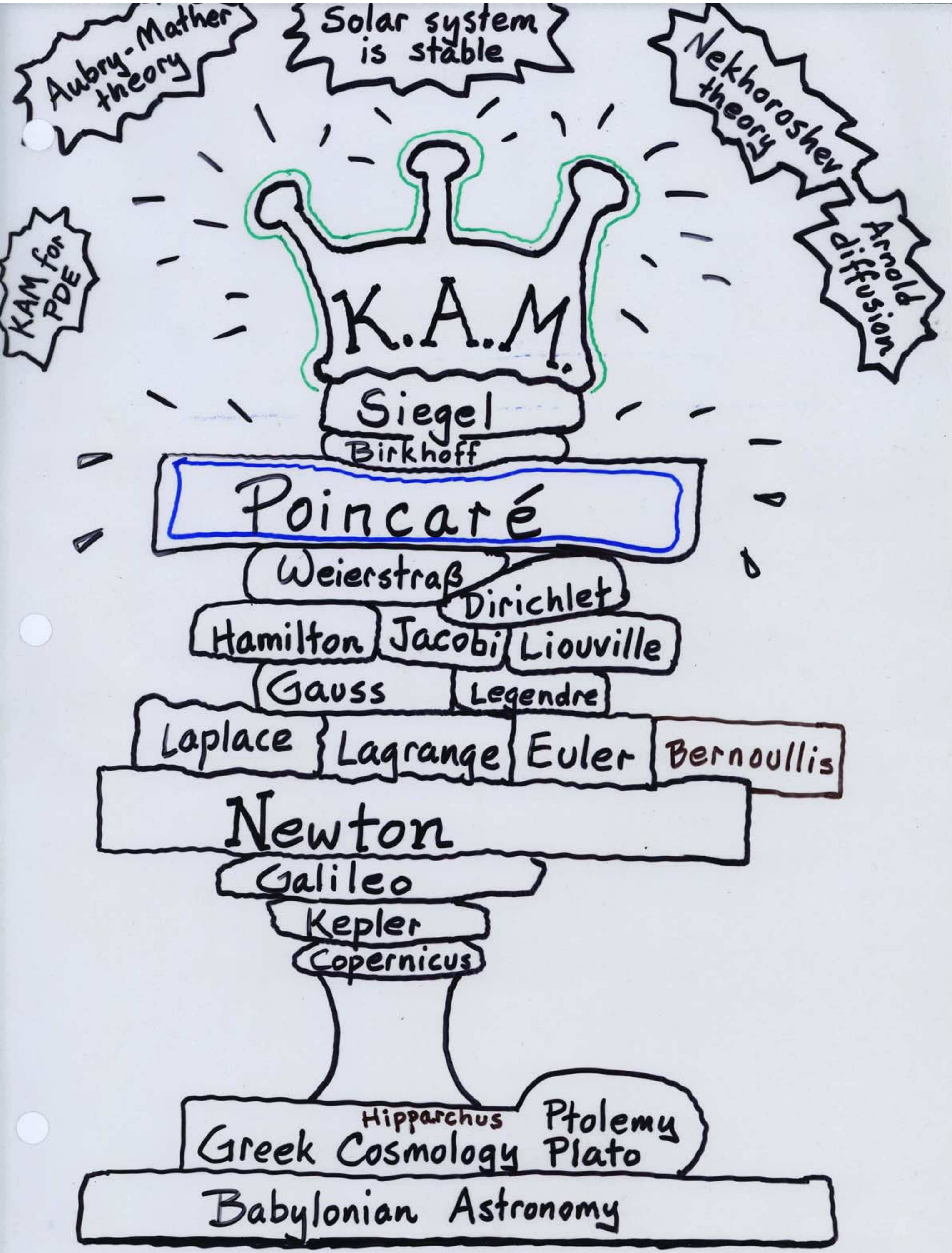
- KAM theory represents a (mathematical) revolution [or "paradigm shift"] in classical mechanics comparable to the revolutions that spawned relativity and quantum theory
- It would've surprised even Poincaré (and pleased Weierstraß)

CONTRA

- It's very technical ("hard analyis")
- Hamiltonian systems are not good mathematical models of physical systems
- The "threshold of validity" (ε) for KAM theory is absurdly small in applications
- KAM theory is over-rated, over-sold, over-romanticized

Come si vede, la meccanica moderna ritorna al modello ideale dei moti circolari uniformi, con la sola differenza che i cerchi non stanno qui nello spazio ordinario, o spazio delle configurazioni, ma in uno spazio più astratto.

—G. Benettin, **Moti** ordinati e moti caotici (Archimede v. 4, 2001)


Aubry-Mather theory
Solar system is stable
Nekhoroshev theory
KAM for PDE
Arnold diffusion
K.A.M.
Siegel
Birkhoff
Poincaré
Weierstraß
Dirichlet
Hamilton
Jacobi
Liouville
Gauss
Legendre
Laplace
Lagrange
Euler
Bernoullis
Newton
Galileo
Kepler
Copernicus
Hipparchus
Ptolemy
Greek Cosmology
Plato
Babylonian Astronomy

Nekhoroshev 1971, 1977, 1979

PROTOTYPE NEKHOROSHEV THEOREM

Let (I, θ) be action-angle variables for the **analytic** completely integrable Hamiltonian (h, M). Assume h is **steep** (or **convex**, or **quasiconvex**).

Then there exist a **nice** set U of initial I-values and (a **small**) $\varepsilon_0 > 0$ such that whenever $I(0) \in U$, and $P = P(I, \theta)$ is an analytic perturbation of size

$$|P|_{U \times T^n} = \varepsilon < \varepsilon_0,$$

the actions $I = I(t)$ of the perturbed Hamiltonian $H(I, \theta) = h(I) + P(I, \theta)$ satisfy $$|I(t) - I(0)| \le \varepsilon^a$$

on the (exponentially long) time interval

$$0 \le t \le \exp(C\varepsilon^{-b})$$

$(a, b, C > 0)$.