Collective Beam-Beam: Theory and Simulation

Andrey Sobol

Contents

- Strong-Strong Beam-Beam Model and the results from paper "*A new model for the collective beam-beam interaction*" by J. A. Ellison and M. Vogt.

- Calculating eigen modes. Difficulties of numerical solution related to the type of the problem (3^{rd} kind integral equation, not compactness of the operator.)

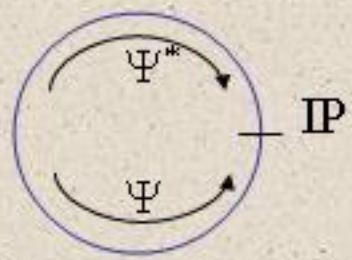
- Direct simulation of the density evolution. WMPT vs. PF approach. Parallel code. Numerical results.

Steps

- Rewrite in slow varying coordinates;
- Using averaging for the map, obtain continuous-time Vlasov equation for $\Psi(t,v)$;
- Linearize this equation around equilibrium,
- Obtain integro-differential equation for π and σ-modes;
- Using Laplas transformation, obtain an integral equation;
- Make Fourier transformation in angle and obtain the system of decoupled integral equation.
- Study the stability by analyzing the spectrum of the integral operator.

Model

- Strong-Strong Beam interaction
- Study density evolution in 2dof case
- One interaction point. Linear lattice.
- Two counter-rotating beams collide at IP.



The beams have densities

$$\psi_n(x, p_x, y, p_y), \quad \psi_n^*(x, p_x, y, p_y),$$

Evolution Equation

The position of the particle in 4d phase space before the interaction point on n-th turn

$$u_n = \left(x, p_x, y, p_y\right)^T$$

changes according to

$$u_{n+1} = M\left(u_n + \zeta \begin{pmatrix} 0 \\ V_x(x_n, y_n) \\ 0 \\ V_y(x_n, y_n) \end{pmatrix} \right)$$

Evolution Equation ★

The potential can be calculated using electrostatic formula in the frame of the moving beam.

$$V(\psi^*)(x,y) = \int_{\Re^2} G(x-x', y-y')\rho^*(x',y')dx'dy'$$

$$G(x,y) = -\ln(\sqrt{x^2+y^2}),$$

The potential in the frame of the kicked beam is calculated by multiplying by

$$\zeta \approx \frac{2N^*}{\gamma}\frac{qq^*}{4\pi\varepsilon_0 mc^2},$$

★ The discussion of the evolution equation was taken from "A new Model for the Collective Beam-Beam Interaction" by J.A. Ellison, M. Vogt

Equation in Slow-Varying Coordinates

Transformation to slow varying variables

$$u_n = R^n \mathrm{v}_n$$

leads to

$$\mathrm{v}_{n+1} = \mathrm{v}_n + \zeta J_4 \nabla_{\mathrm{v}} H(\Psi_n^*)(\mathrm{v}_n, n),$$

where

$$H(\Phi)(\mathrm{v},\mathrm{n}) = \int G(\sqrt{2\beta_x J_x}\sin(2\pi n \nu_x + \Theta_x) - \sqrt{2\beta J_x'}\sin(2\pi n \nu_x^* + \Theta_x'), x \to y)\Phi(\mathrm{v}')d\mathrm{v}'.$$

Averaging leads to

$$\mathrm{v}_{n+1} = \mathrm{v}_n + \zeta J_4 \nabla_{\mathrm{v}} \bar{H}(\Psi_n^*)(\mathrm{v}_n) + O(\zeta^2).$$

This corresponds to the equations

$$\partial_t \Psi + \zeta\{\Psi, \bar{H}(\Psi^*)\} = 0, \quad \partial_t \Psi^* + \zeta^*\{\Psi^*, \bar{H}^*(\Psi)\} = 0.$$

- Linearize about equilibrium

$$\Psi(v,t) = \Psi_e(v) + \Psi_1(v,t)$$

- Study π and σ-modes:

$$f = \Psi_1 \pm \Psi_1^*$$

- We are looking for the solutions in form.

$$g(J,t) = e^{i\omega t}\hat{g}(J,\omega)$$

- Azimuthal Fourier modes are the solutions of the uncoupled integral equation of the third kind.

$$(\omega - k\cdot\Omega_e(J))\hat{g}(J,\omega) - \eta\int K_k(J,J')\hat{g}(J',\omega)dJ' = 0$$

Difficulties of the numerical solution of the 3rd kind integral equation

- This kind of equation has been studied by Warnock (*"Linear Integral Equations of the third kind"*, G. R. Bart R.L. Warnock, SIAM, vol. 4, No. 4, Nov. 1973).

- This integral operator has continuous spectrum with eigenfunctions being delta functions. It makes it hard to present numerically. The operator is not compact therefore discretization of these equation doesn't converge.

Regularizing the Integral Equation

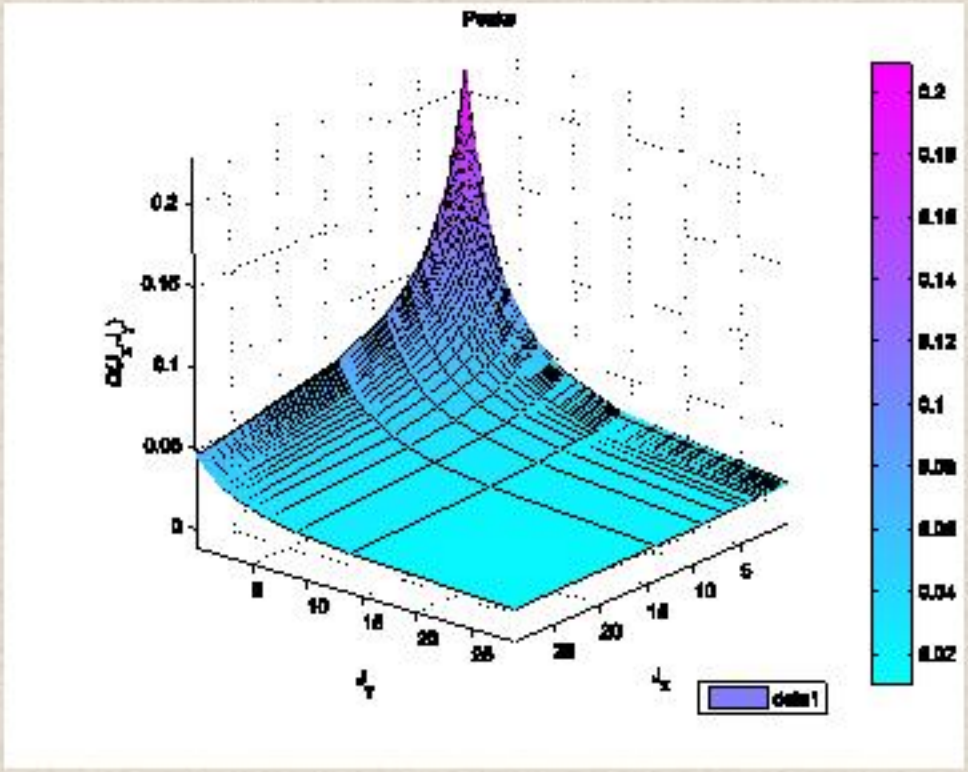
- The equation must be transformed. For the 1dof case this was made in "*Linear Vlasov Analysis for stability of a bunched beam*" by R. Warnock, G. Stupakov, M. Venturini, Ellison and *"Nonsingular integral Equation for stability of bunched beam"* by R. Warnock, M. Venturini, and J. Ellison.

$$\hat{g}(J,\omega) - \eta \int \frac{\hat{g}(J',\omega)}{(\omega - k \cdot \Omega_e(J))} K_k(J,J') dJ' = 0$$

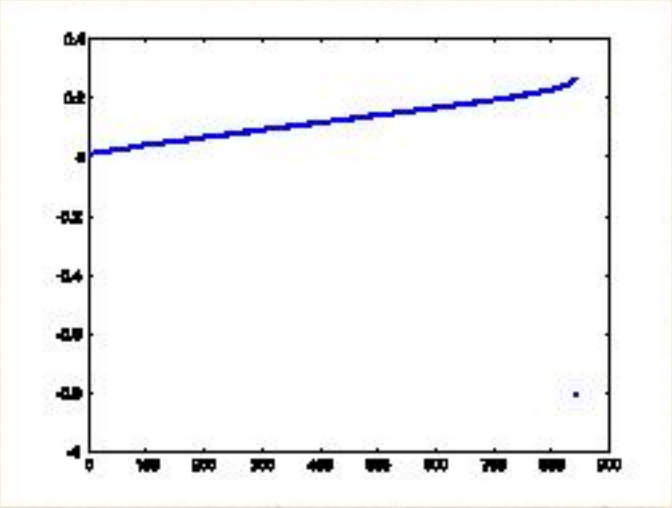
- We are calculating eigenvalues and eigenfunctions for 2dof case

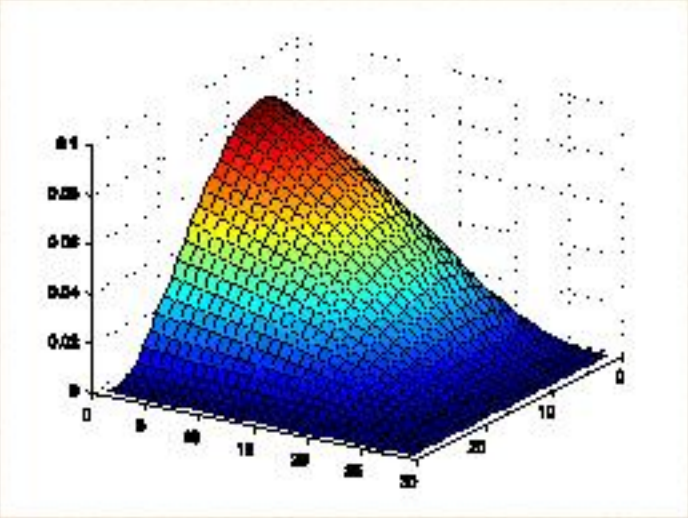
2dof case discretization

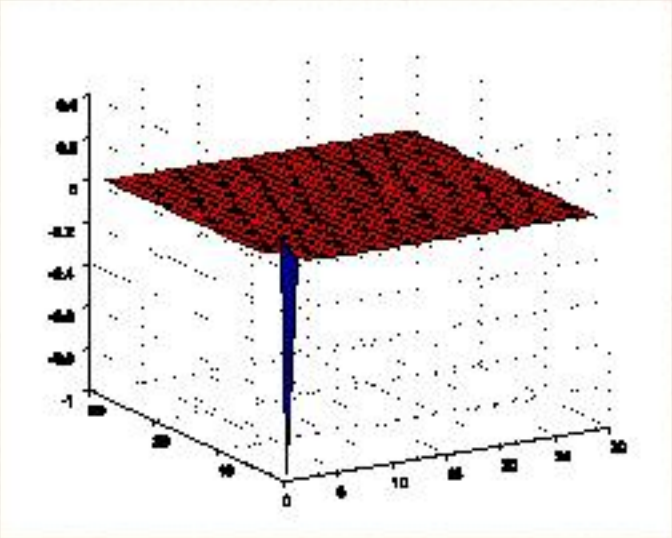
- To calculate the coefficients of this integral equation we have to deal with high dimensionality of the problem and the necessity of taking improper integrals.

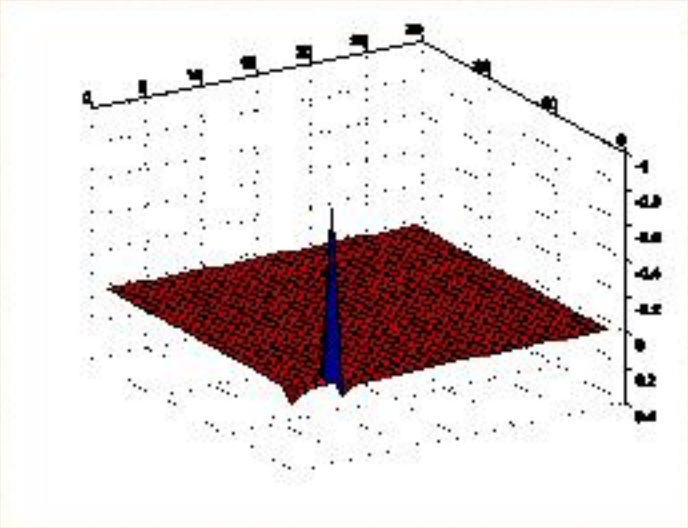


σ-mode

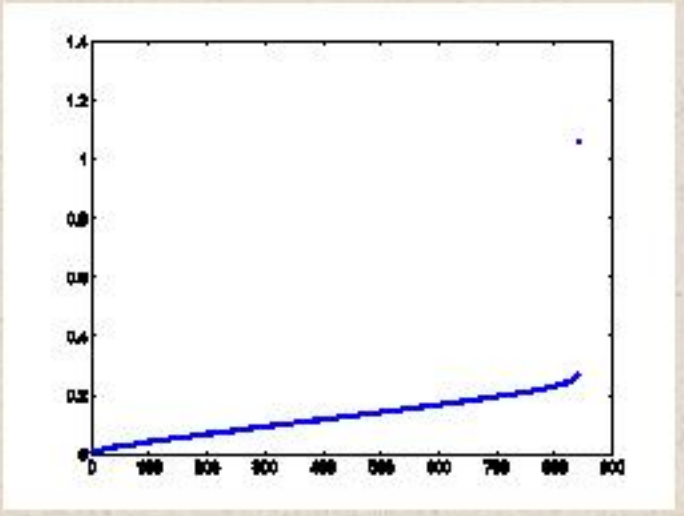


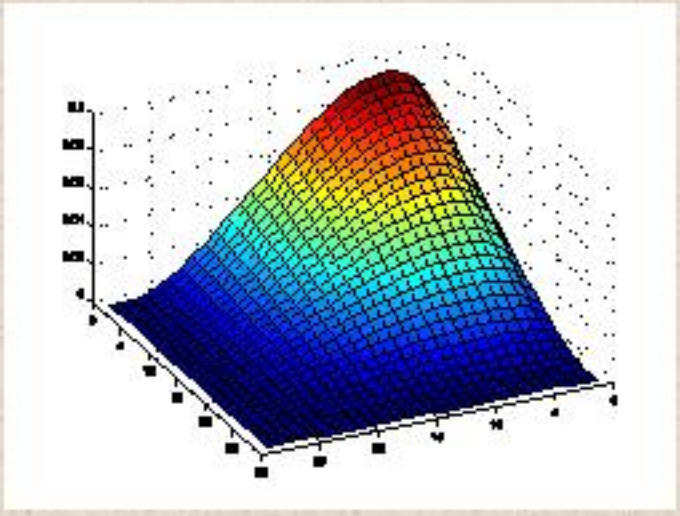


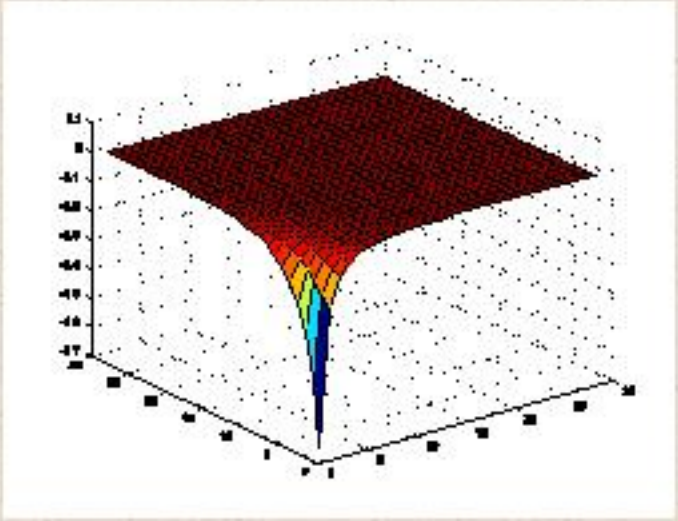


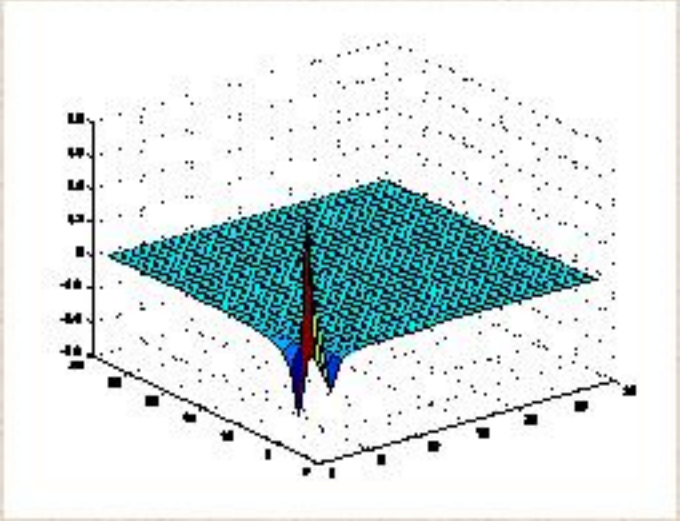


π-mode









Analytical result

- Warnock suggested that the transformed equation is compact in some norm. I think I proved that it is indeed compact in

$$\| f \|_a = \sup \left| \frac{f(x) - f(a)}{x - a} \right| + | f(a) |$$

- This is an important step to find a convergent discretisation of the equation.

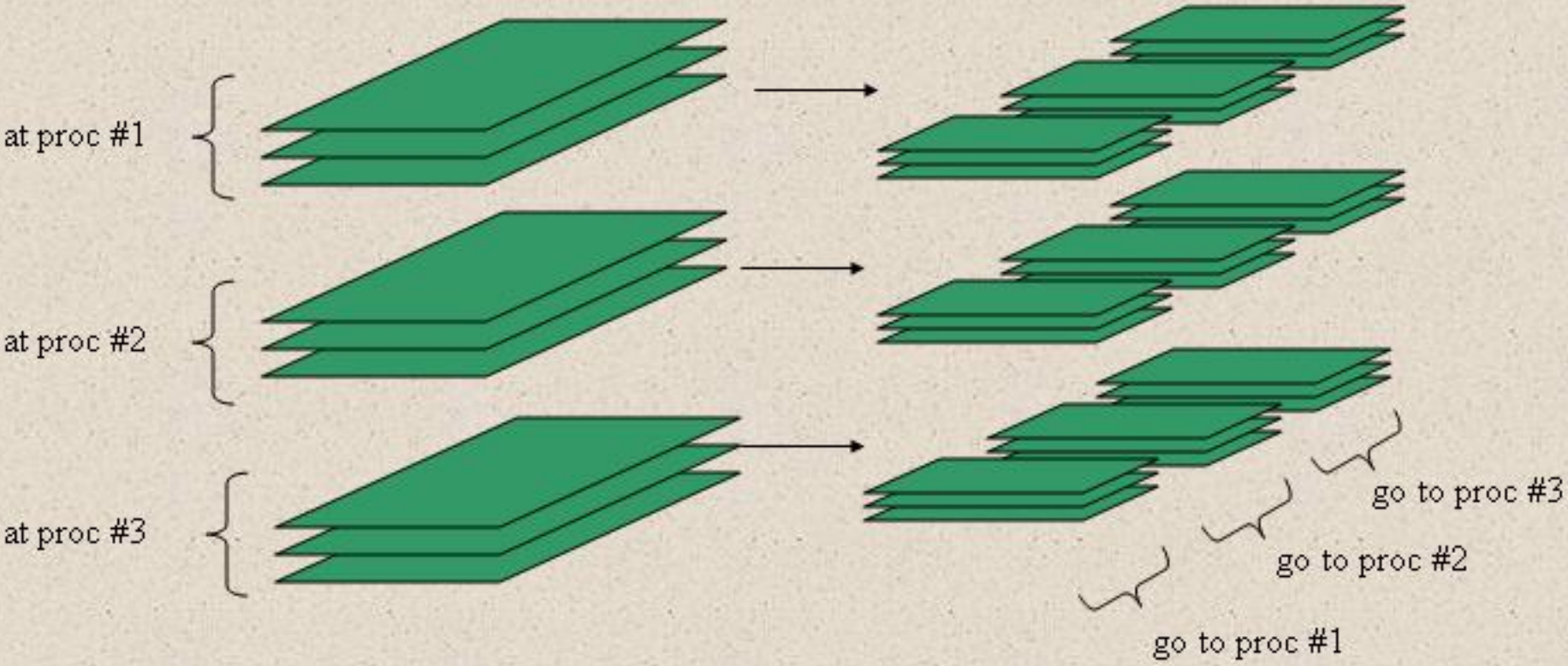
Direct Simulations

- Macroparticle approach
- Weighted macroparticle approach. WMPT code by Mathias Vogt.
- PF method: 1 d.o.f. is done by Bob Warnock and Mathias Vogt.
- 2 dof. Simulation in 4 dimensional phase space. Need of parallel computations.

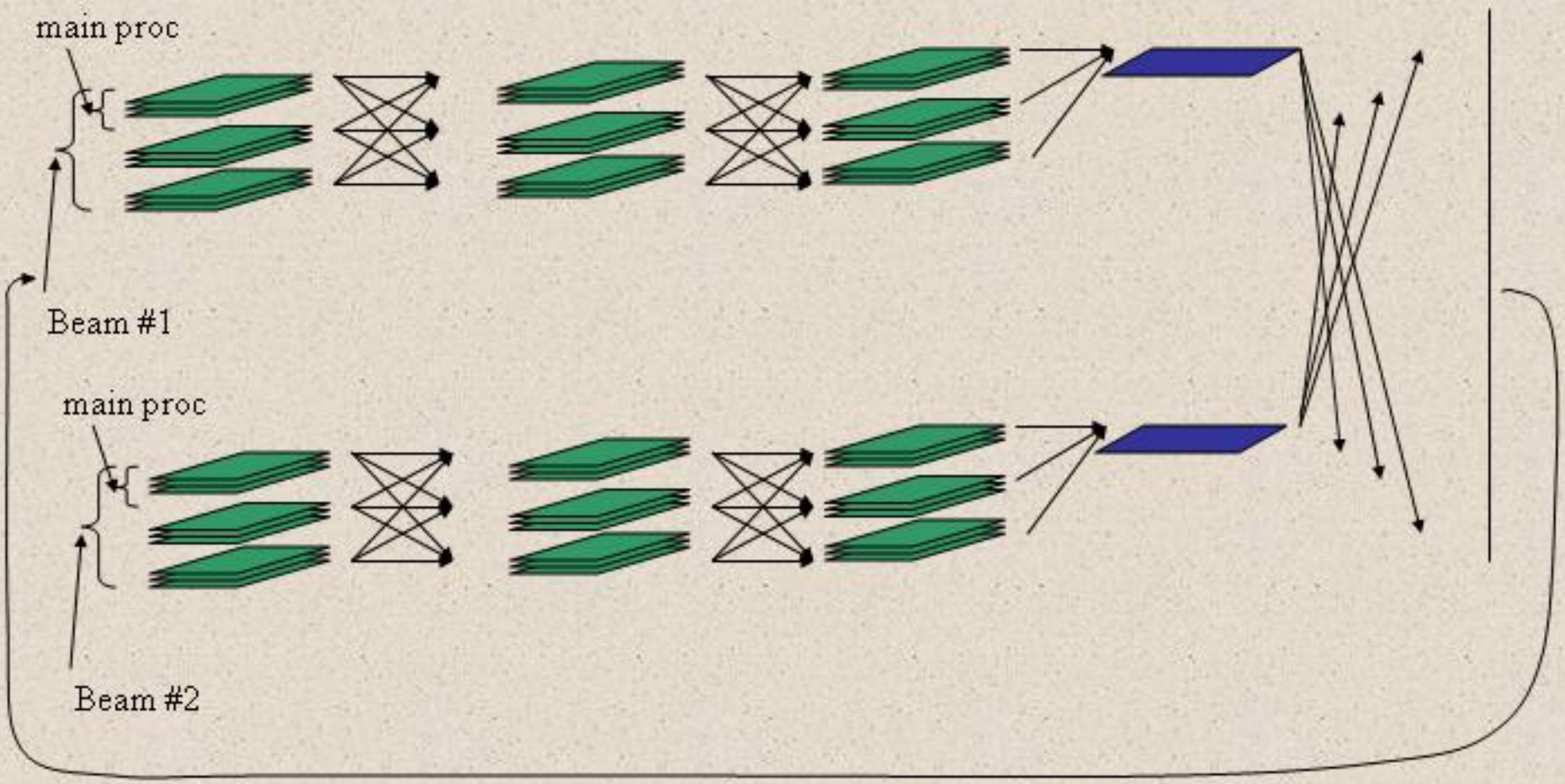
$$u_{n+1} = R \circ K[\Psi_n^*](u_n), \qquad u_{n+1}^* = R \circ K[\Psi_n](u_n^*),$$

$$\Psi_{n+1} = \Psi_n \circ K^{-1}[\Psi_n^*] \circ R^{-1}, \quad \Psi_{n+1}^* = \Psi_n \circ K^{-1}[\Psi_n^*] \circ R^{-1},$$

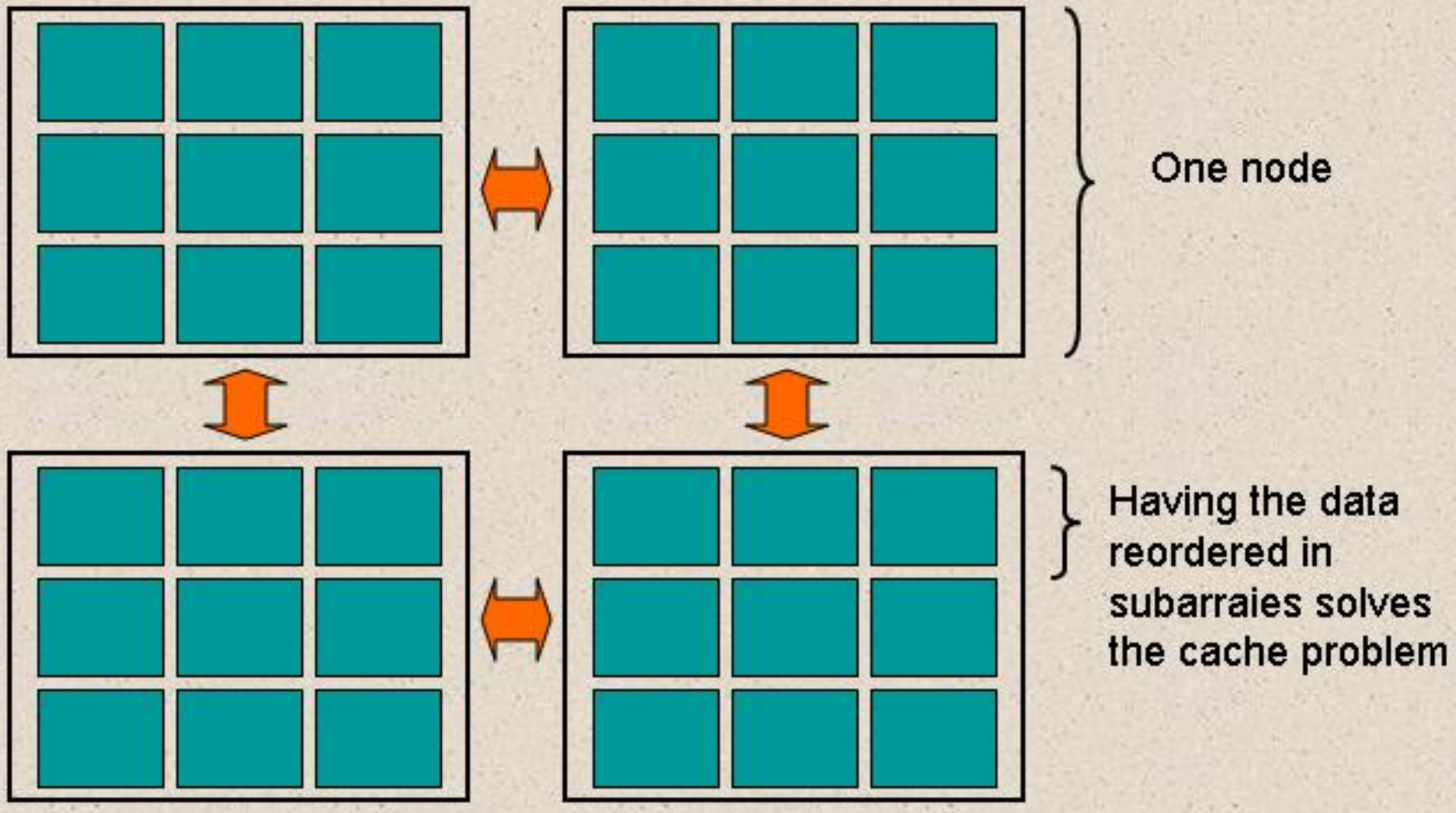
Transformation of the Data



Transformation of the Data



Density evolution in slow varying coordinates in Slow Varying coordinates



First results of parallel simulations

- Round Gaussian Source approximation
- Finding kick by solving Poisson equation.