Intra-Beam Scattering and the Coasting beam in HERA-p.

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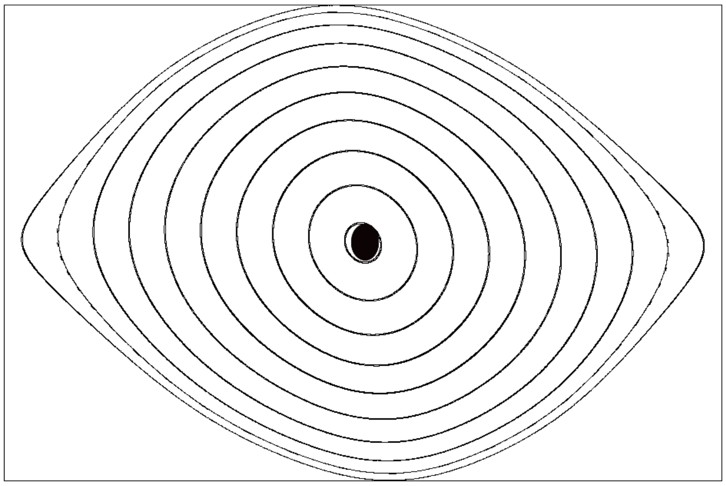
Overview

- The coasting beam and its potential sources
- Simulation of space charge phenomena
- Models of small random perturbations arising from intra-beam scattering
- Modeling with Markov chains
- Some results and future lines of work

The Coasting Beam



Single particle longitudinal motion - Hamiltonian system with $H(\varphi, p)$ and thus stable
Coasting beam - particles escaped from the stable RF bucket and not yet lost in the transverse direction.

Perturbations:

$$\dot{\varphi} = \frac{\partial H}{\partial p} + \varepsilon \xi(t)$$

$$\dot{p} = -\frac{\partial H}{\partial \varphi} + \varepsilon \nu(t)$$

Noticable influence may be seen on time scales ε^{-1}, including separatrix crossing.

2 major problems

1. Understanding mechanisms of production
2. Influence on experiments, registration and collimation

Sources of perturbations

- Single particle effects
  - Optics
  - RF noise
  - feedback
- Collective effects
  - Wake Fields
  - Space charge
    1. Collective part
    2. Random part - scattering

Self consistent equations for the space charge

$$H(p,q,t)$$

Liouville

$$\frac{\partial f}{\partial t} + (\frac{dH}{dp}, \frac{df}{dq}) - (\frac{dH}{dq}, \frac{df}{dp}) = 0$$

Maxwell-Vlasov

$$\frac{\partial f}{\partial t} + (\frac{dH}{dp}, \frac{df}{dq}) + (-\frac{dH}{dq} + E(q) + \frac{1}{c}[v(p,q) \times H], \frac{df}{dp}) = 0$$

$$rot H = j + \frac{\partial E}{\partial t}$$

$$rot E = -\frac{\partial H}{\partial t}$$

$$div H = 0$$

$$div E = \rho$$

'Maxwell-Vlasov-Boltzmann'

$$\frac{\partial f}{\partial t} + (\frac{dH}{dp}, \frac{df}{dq}) + (-\frac{dH}{dq} + E(q) + \frac{1}{c}[v(p,q) \times H], \frac{df}{dp}) = C(f,f)$$

Through away the Maxwell equations and average over the ring

$$\frac{\partial f}{\partial t} = C(f,f)$$

Still no chance to apply

Emittance growth

(Bjorken/Mtingwa, Piwinski)

If distributions are assumed permanently Gaussian

$$\rho(x,p) = \frac{1}{\Gamma(t)} e^{-S(X,\Gamma(t))}$$

then changes of statistical distribution functions (incl. ρ itself) can be computed from

$$\frac{df(X)}{dt} = C(X)$$

$$C = \int dX' \phi(X') \rho(X') \rho(X) [f(X') - f(X)]$$

If one substitutes ρ and averages over X one normally can estimate $< f(X) >$ analitycally after some effort.

In other words, emittance growth $\epsilon(t)$ can be computed in this way.

This approach may lead to wrong estimates of the escape rate since

1. In the tails the 'kinetic parameters' (basicly the free path length) sufficiently deviate from the mean and so does behaviour of particles in them.

2. The distribution may considerably deviate from Gaussian after some time

References:

1. Piwinski, 'Intra-beam scattering' family of papers
2. Bjorken, Mtingwa , Intra-Beam scattering (FNAL 1982)
3. Any good book on kinetic theory

Single particle motion as a randomly perturbed Hamiltonian system

A numerical solution of a Boltzmann-type equation is normally done either employing linearization of the integral operator or with a Monte-Carlo simulation. Both are equivalent to treating the evolution of the distribution as that of an ensemble of particles under influence of some random perturbation

A single-particle motion within the bunch can be viewed as Markov random perturbation under following assuptions

1. Molecular (proton) chaos
2. The distribution doesn't change on time scales at which the process is considered Markov.

Markov process (process without aftereffect)

$$P(x(t) \in A | x(s_i) \in B_i) = P(x(t) \in A | x(s_n) \in B_n)$$

$$s_1 \leqslant \ldots \leqslant s_n \leqslant t$$

Infinitesimal generator

$$AV(x) = \lim_{h \to 0} \frac{E^x V(x+h) - V(x)}{h}$$

Sufficiently regular Markov processes are

1. Jump processes
$$Af(x) = \int K(x,y) f(y) dy$$
2. Diffusions
$$Af(x) = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} f(x)) + \frac{\partial}{\partial x}(a(x) f(x))$$
normally easier to solve.

Evolution of the distribution function

$$\frac{\partial f(x,t)}{\partial t} = Af(x,t)$$

The random scattering is a jump process since the velocities are changed extremely quickly during a collision.

On the diffusion approximation

A diffusion process has continuous path and thus physically differs from the scattering process. However, one can sometimes find a diffusion process whose distribution evolves approximately like that of a jump process.

This can be done

1. Formally by expanding the integral operator into series and throwing out higher order derivatives
2. By observing that if the increment of a jump process is approximately gaussian

$$\rho(x(t+\Delta t)-x(t)) = N(\Delta t a(x), \sigma\sqrt{(}\Delta t))$$

then there exists a diffusion process having the same increment with drift $a(x)$ and diffusion $\sigma(x)$

This may lead to errors when

1. $\frac{d}{dx}\sigma(x)$ is large
2. $\sigma(x) = 0$ or small in some subdomains of the domain of interest - the diffusion process will never reach some regions which the jump process reaches.

This is the case for the scattering -for particles near the separatrix the free path is extremely large.

A diffusion approximation in the sence of distributional convergence is not possible

Invariant measure

Jump process in 6D - still too complex - reduce dimensionality

Assume the motion is uncoupled ,i.e

$$H = H_z + H_x + H_y$$

for 1D motion let $H(p,q) = \frac{p^2}{2} + U(q)$

$$\psi = Arctg\frac{p}{q}$$

$$H = \frac{p^2}{2} + U(q)$$

It is nonsingular and its Jacobian is

$$|J| = \begin{vmatrix} \psi_p & \psi_q \\ H_p & H_q \end{vmatrix} = \begin{vmatrix} \frac{q}{p^2+q^2} & -\frac{p}{p^2+q^2} \\ 2\lambda p & U'(q) \end{vmatrix} = \frac{1}{p^2+q^2}(U'(q)q + 2\lambda p^2) \tag{1}$$

$$f(p,q) = \hat{f}(H,\psi)|J|$$
$$\hat{f}(H,\psi)dHd\psi = \rho(H)\mu_H(\psi)dHd\psi$$

$$\frac{dt(d\psi)}{T(H)} = \mu_H(\psi)d\psi \tag{2}$$

where $T(H)$ is the oscillation period,

$$T(H) = 2\int_{-\sqrt{2H}}^{\sqrt{2H}} \frac{dy}{\sqrt{2H - 2U(y)}}$$

$$p = \sqrt{2H - 2U(q)}, \quad p'(q) = -\frac{U'(q)}{\sqrt{2H - 2U(q)}} = \tan\varphi$$

$$\varphi = -Arctan\left[\frac{U'(q)}{p}\right]$$

$$dq = pdt$$

$$dq = rd\psi cos\varphi = \sqrt{p^2+q^2}\cos(Arctan[\frac{U'(q)}{p}])d\psi = \sqrt{p^2+q^2}\frac{1}{\sqrt{\frac{U'^2(q)}{p^2}+1}}d\psi$$

$$\mu_H(\psi) = \sqrt{p^2+q^2}\frac{1}{\sqrt{U'^2(q)+p^2}}\frac{1}{T}$$

$$f(p,q) = \sqrt{\frac{1}{(p^2+q^2)(p^2+U'^2(q))}}\left(p^2+qU'(q)\right)\frac{1}{T}\rho(H(p,q)) \quad (3)$$

For harmonic oscillator and Maxwell-Boltzmann distribution in energy

$$\rho(H) = \alpha e^{-\alpha H}, \quad U(q) = \frac{q^2}{2}$$

$$f(p,q) = \frac{1}{2\pi\alpha^{-1}}e^{-0.5\alpha(p^2+q^2)}$$

$$f(p) = \int_{-\infty}^{\infty} f(p,q)dq = \frac{1}{\sqrt{2\pi\alpha^{-1}}}e^{-\frac{p^2}{2\alpha^{-1}}}$$

Under assumption that the bunch is sufficiently rarified all kinetic parameters are defined by mean values of the density along the path and thus by H.

For small times a single particle motion can be described by 3 independent 1D random walks of H_z, H_x, H_y

Jump processes again

1. Continuous state space
   $f(H)$, $H \in [0, H_{max}]$

$$\frac{\partial f}{\partial t} = \int_{-\infty}^{\infty} K(x,y) f(y,t) dy$$

   K(x,y) is computed numerically
   f may be found from an appropriate difference scheme

2. Discrete state space
   $x_i = P(H_i < H \leqslant H_{i+1})$
   there exists certain matrix $P(t - t_0)$ so that $x^T(t) = x^T(t_0) P(t - t_0)$
   $P(t)$ satisfies some system of linear differential equations

The choice between the two approaches is a matter of taste and computational costs

Markov chains

The Markov chain is given by the space of states Ω and the transition probabilities in state i to state j in time t $p_{ij}(t)$ satisfying

1. $0 \leqslant p_{ij}(t) \leqslant 1$
2. $\sum_0^n p_{ij}(t) = 1$
3. $\forall t > 0, s > 0$ $p_{ij}(t+s) = \sum_{k=0}^n p_{ik}(t)p_{kj}(s)$ (Chapman-Kolmogorov equation)
4. $\lim_{t\downarrow 0} p_{ij}(t) = \delta_{ij}$ (stochastic continuity)

Then it can be shown that the following limits exist

$$a_{ij} = \lim_{t\downarrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h}$$

If $i \neq j$ then the limits are always bounded. Therefore, the transition probabilities satisfy the Kolmogorov's system of equations

$$\frac{d}{dt} p_{ij}(t) = \sum a_{ik} p_{kj}(t) \tag{4}$$

the system possesses a unique solution if $a_{ij} \geqslant 0$, $a_{ii} \leqslant 0$ $\sum a_{ij} = 0$ $sup\,|a_i i| < \infty$ and the solution satisfies the initial conditions $p_{ik}(0+) = \delta_{ik}$

Let

$$p_{ii}(h) = 1 - \varepsilon b_{ii} h + O(h^2)$$

$$p_{ij}(h) = \varepsilon b_{ij} h + O(h^2), \quad i \neq j$$

then

$$a_{ii} = -\varepsilon b_{ii} \leqslant 0$$

$$a_{ij} = \varepsilon b_{ij} \geqslant 0$$

denote $x_1 = p_{10}, \ldots, x_{n+1} = p_{1n}$, $x_{n+2} = p_{20}, \ldots, x_{2(n+1)} = p_{2n}$, ...
$x_{(n-1)(n+1)} = p_{n1}, \ldots, x_{n(n+1)} = p_{nn}$,

$$\dot{x}(t) = \mathcal{A}x(t)$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{1n} \\ \dots & \dots & \dots \\ \mathcal{A}_{n1} & \dots & \mathcal{A}_{nn} \end{pmatrix} \tag{5}$$

and $\mathcal{A}_{ij} = diag\{a_{ij}\} \in \mathbb{R}^n$ is a diagonal matrix with diagonal elements all equal.
Going to a slower time scale $\tau = \varepsilon t$ we obtain

$$\dot{x}(\tau) = \frac{1}{\varepsilon}\mathcal{A}x(\tau) = \hat{\mathcal{A}}x(\tau)$$

where

$$\hat{\mathcal{A}} = \begin{pmatrix} \hat{\mathcal{A}_{11}} & \dots & \hat{\mathcal{A}_{1n}} \\ \dots & \dots & \dots \\ \hat{\mathcal{A}_{n1}} & \dots & \hat{\mathcal{A}_{nn}} \end{pmatrix} \tag{6}$$

$\hat{\mathcal{A}}_{ii} = diag\{-b_{ii}\}$, $\hat{\mathcal{A}}_{ij} = diag\{b_{ij}\}$, $b_{ij} \geqslant 0$
The Kolmogorov system is a linear system and thus there are numerous ways to solve it, f.e. the solution can be written as

$$x = \exp\{At\}x(0)$$

If the initial probability distribution of the chain is ξ_0 then after time t
$\xi(t) = \xi_0^T P(\varepsilon t)$

Markov chains with 'self-consistent' transition probabilities

Let $P(x,t)$ be a function from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^{n \times n}$, $P(x,0) = E$,
$[0,T]$ is partitioned into $\{h_i\}$.

$$\begin{array}{lcll} x_\tau & = & x_0^T P(x_0,\tau) & ,0 \leqslant \tau \leqslant h_1 \\ x_\tau & = & x_{h_1}^T P(x_{h_1},\tau) & ,h_1 \leqslant \tau \leqslant h_2 \\ \ldots & \ldots & \ldots & \ldots \\ x_\tau & = & x_{h_i}^T P(x_{h_i},\tau) & ,h_i \leqslant \tau \leqslant h_{i+1} \\ \ldots & \ldots & \ldots & \ldots \end{array} \tag{7}$$

1. Under assumption of sufficient regularity of $P(x,\tau)$. there exists a constant C such that $\|x_\tau - y_\tau\| < C\varepsilon$ on $\tau \in [0,T]$ whenever $\|x_0 - y_0\| < \varepsilon$ for bounded time intervals $[0,T]$.

2. the sequence of functions x_τ converges to some limit $x_\tau^\star$ as $max\,|h_{i+1} - h_i| \to 0$ for bounded T

3. If we appropriately tend the number of states of the Markov chain to ∞ then the x_τ will converge in some sence to the solution of the integral equation for a corresponding jump process with continuous state space.

Infinitesimal generator

$$a_{ij} = \lim_{h \downarrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h}$$

$Q = \{q_{ij}\}$ - matrix of transition probabilities between the states in a single collision

$\hat{\gamma}_i(\tau) = \gamma_i \tau + O(\tau^2)$ - probability of a collision of a particle in state i

transition matrix for small times

$$\{p_{ij}\} = \begin{pmatrix} q_{11}\gamma_1\tau + (1-\gamma_1\tau) & q_{12}\gamma_1\tau & \dots & q_{1n}\gamma_1 \\ q_{12}\gamma_2\tau & q_{22}\gamma_2\tau + 1 - \gamma_2\tau & \dots & q_{2n}\gamma_2 \\ \dots & \dots & \dots & \dots \\ q_{n1}\gamma_n\tau & q_{n2}\gamma_n\tau & \dots & q_{nn}\gamma_n + (1-\gamma_n\tau) \end{pmatrix} + O(\tau^2) \tag{8}$$

1. Invariant changes in a single collision

   The momentum change in a collision is approximately

$$\delta p_{12} = \pm\frac{p}{2} \begin{pmatrix} c1\cos\phi\sin\psi + c2(\cos\psi - 1) \\ (c3\sin\phi - c4cos\phi)\sin\psi + c5(\cos\psi - 1) \\ (c6\sin\phi - c7cos\phi)\sin\psi + c8(\cos\psi - 1) \end{pmatrix} \tag{9}$$

   where ϕ - uniformly distributed polar angle, ψ- azimuthal angle depending on the impact parameter

   From δp change of invariants can be computed

   Transversal

$$I_{x,y} = \gamma_{x,y}u_{x,y} + 2\alpha_{x,y}u_{x,y}u'_{x,y} + \beta_{x,y}u'_{x,y}$$

$$\frac{\delta p_{x,y}}{p} = \delta u'_{x,y}$$

   Longitudinal

$$I_z = \frac{\delta^2 p}{p} + \Omega(1 - \cos\varphi)$$

2. Collision frequency

$$\gamma(t) = N_p \pi r_0^2 t < vf(q) >$$

Q and γ can be computed by Monte-Carlo trials.

Computational costs

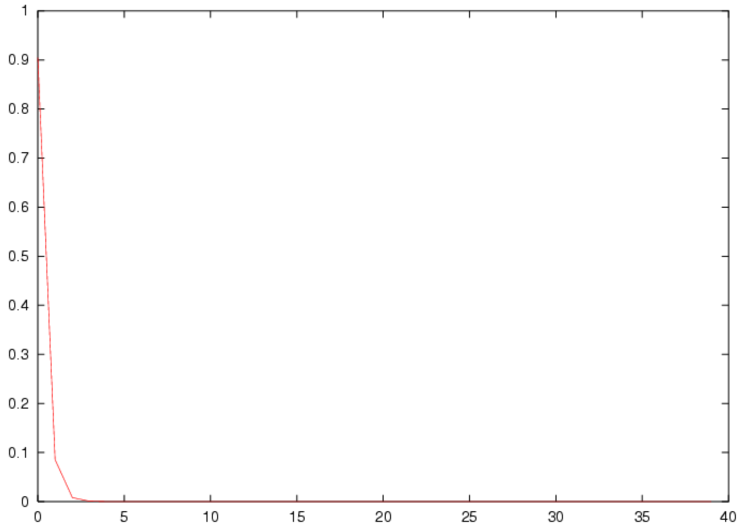
number of states of the Markov chain	N
dimension of the transition probability matrix	$N \times N$
dimension of the infinitesimal generator matrix	$N^2 \times N^2$
number of Monte-Carlo trials for computing Q	$K_1 N^2$
number of Monte-Carlo trials for computing γ	$K_2 N$
number of operations for solving the Kolmogorov system	??? not less than $K_3 N^6$

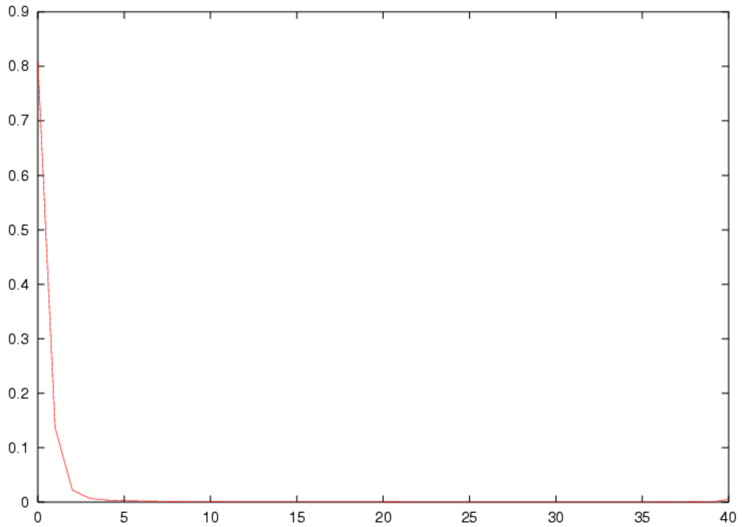
the bottleneck is of course the Kolmogorov system

Some results

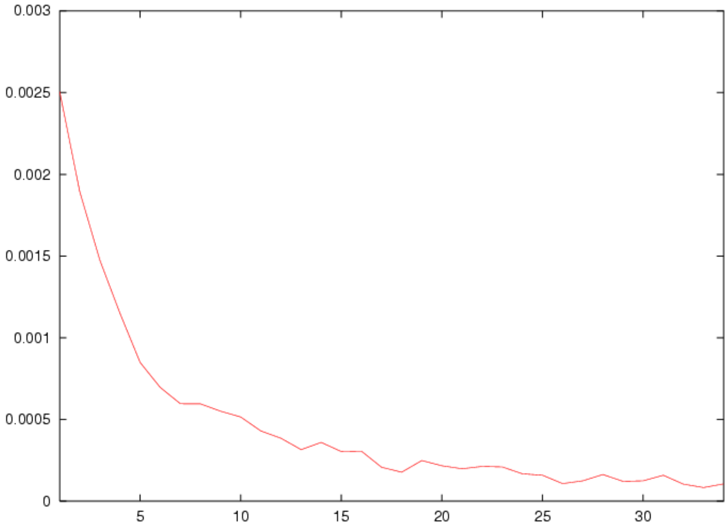
$T = 10hours$

- Evolution of distribution function

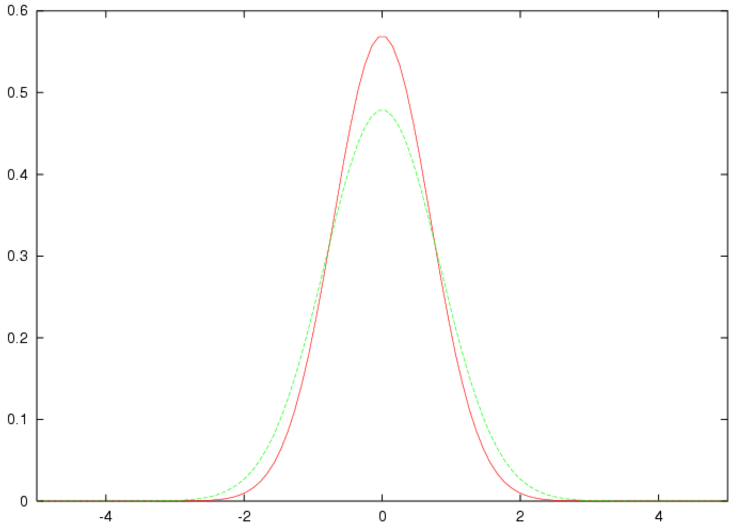




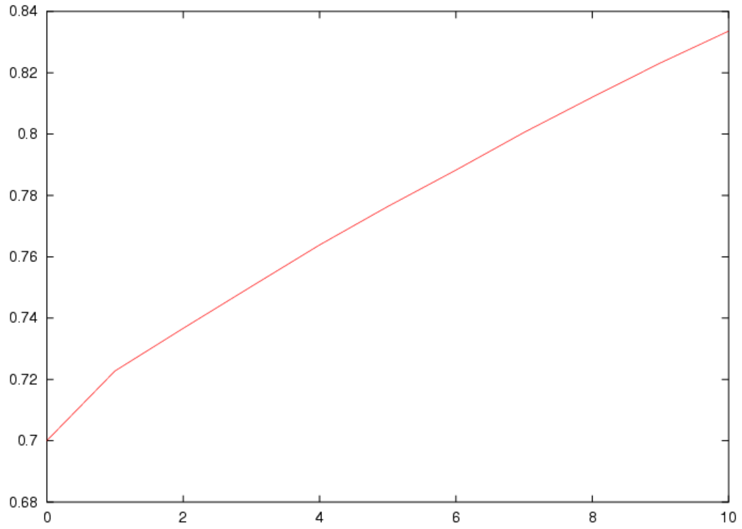
- Tails



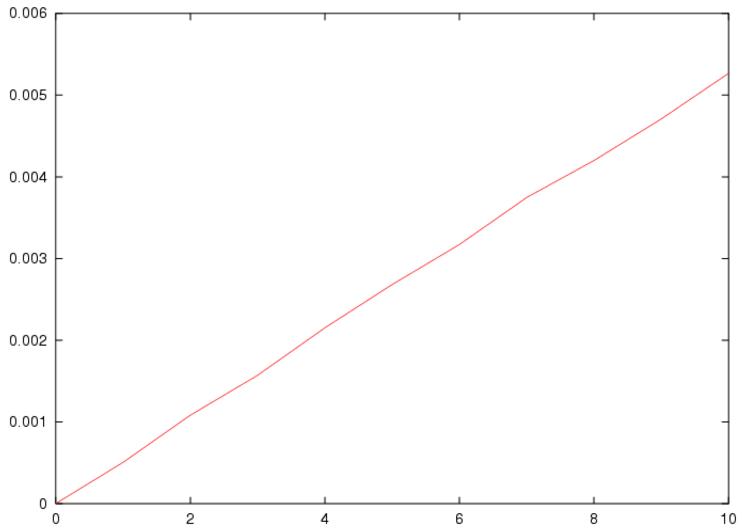
- Gaussian fit



- Evolution of the longitudinal σ in ns.



- Particle losses



Possible future lines of work

- IBS
  - Code optimization
  - Jump process with continuous phase space
- Random longitudinal perturbations
- What can be done if the coasting beam is unavoidable

The End.